Mechanics 1 (AH)

M1.1 Motion in a straight line

M1.1.1 know the meaning of position, displacement, velocity, acceleration, uniform speed, uniform acceleration, scalar quantity, vector quantity

Concepts of position, velocity and acceleration should be introduced using vectors.

Candidates should be very aware of the distinction between scalar and vector quantities, particularly in the case of speed and velocity.

M1.1.2 draw, interpret and use distance/time, velocity/time and acceleration/time graphs

Candidates should be able to draw these graphs from numerical or graphical data.

M1.1.3 know that the area under a velocity/time graph represents the distance travelled

M1.1.4 know the rates of change $v = \frac{dx}{dt} = \dot{x}$ and $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \dot{v} = \ddot{x}$

Candidates should be familiar with the dot notation for differentiation with respect to time.

M1.1.5 derive, by calculus methods, and use the equations governing motion in a straight line with constant acceleration, namely:

$v = u + at$, $s = ut + \frac{1}{2}at^2$ and from these,

$v^2 = u^2 + 2as$, $s = (u + v)t/2$

Candidates need to appreciate that these equations are for motion with *constant* acceleration only. The general technique is to use calculus.

M1.1.6 solve analytically problems involving motion in one dimension under constant acceleration, including vertical motion under constant gravity

M1.1.7 solve problems involving motion in one dimension where the acceleration is dependent on time, ie $a = \frac{dv}{dt} = f(t)$

M1.2 Position, velocity and acceleration vectors including relative motion

M1.2.1 know the meaning of the terms relative position, relative velocity and relative acceleration, air speed, ground speed and nearest approach

M1.2.2 be familiar with the notation:

$\mathbf{r}_P$ for the position vector of P

$\mathbf{v}_P = \dot{\mathbf{r}}_P$ for the velocity vector of P

$\mathbf{a}_P = \dot{\mathbf{v}}_P = \ddot{\mathbf{r}}_P$ for the acceleration vector of P

$\overrightarrow{PQ} = \mathbf{r}_Q - \mathbf{r}_P$ for the position vector of Q relative to P

$\mathbf{v}_Q - \mathbf{v}_P = \dot{\mathbf{r}}_Q - \dot{\mathbf{r}}_P$ for the velocity of Q relative to P

$\mathbf{a}_P - \mathbf{a}_Q = \dot{\mathbf{v}}_P - \dot{\mathbf{v}}_Q = \ddot{\mathbf{r}}_P - \ddot{\mathbf{r}}_Q$ for the acceleration of Q relative to P

M1.2.3 resolve vectors into components in two and three dimensions

This requires emphasis.

M1.2.4 differentiate and integrate vector functions of time

M1.2.5 use position, velocity and acceleration vectors and their components in two and three dimensions; these vectors may be functions of time

M1.2.6 apply position, velocity and acceleration vectors to solve practical problems, including problems on the navigation of ships and aircraft and on the effect of winds and currents

Candidates should be able to solve such problems both by using trigonometric calculations in triangles and by vector components.

Solutions by scale drawing would not be accepted

M1.2.7 solve problems involving collision courses and nearest approach

M1.3 Motion of projectiles in a vertical plane

M1.3.1 know the meaning of the terms projectile, velocity and angle of projection, trajectory, time of flight, range and constant gravity

Candidates also require to know how to resolve velocity into its horizontal and vertical components.

M1.3.2 solve the vector equation $\ddot{\mathbf{r}} = -g\mathbf{j}$ to obtain $\mathbf{r}$ in terms of its horizontal and vertical components

The vector approach is particularly recommended.

M1.3.3 obtain and solve the equations of motion $\ddot{x} = 0$, $\ddot{y} = -g$, obtaining expressions for $\dot{x}$, $\dot{y}$, x and y in any particular case

M1.3.4 find the time of flight, greatest height reached and range of a projectile

Only range on the horizontal plane through the point of projection is required.

M1.3.5 find the maximum range of a projectile on a horizontal plane and the angle of projection to achieve this

M1.3.6 find, and use, the equation of the trajectory of a projectile

Candidates should appreciate that this trajectory is a parabola.

M1.3.9 solve problems in two-dimensional motion involving projectiles under a constant gravitational force and neglecting air resistance

Applications from ballistics and sport may be included and vector approaches should be used where appropriate.

M1.4 Force and Newton's laws of motion

M1.4.1 understand the terms mass, force, weight, momentum, balanced and unbalanced forces, resultant force, equilibrium, resistive forces

M1.4.2 know Newton's first and third laws of motion

M1.4.3 resolve forces in two dimensions to find their components

Resolution of velocities, etc. has been covered in previous sections.

M1.4.4 combine forces to find resultant force

M1.4.5 understand the concept of static and dynamic friction and limiting friction

M1.4.6 understand the terms frictional force, normal reaction, coefficient of friction μ, angle of friction λ, and know the equations $F = \mu R$ and $\mu = \tan\lambda$

Balanced, unbalanced forces and equilibrium could arise here.

Candidates should understand that for stationary bodies, $F \leqslant \mu R$.

M1.4.7 solve problems involving a particle or body in equilibrium under the action of certain forces

Forces could include weight, normal reaction, friction, tension in an inelastic string, etc.

M1.4.8 know Newton's second law of motion, that force is the rate of change of momentum, and derive the equation $\mathbf{F} = m\mathbf{a}$

M1.4.9 use this equation to form equations of motion to model practical problems on motion in a straight line

M1.4.10 solve such equations modelling motion in one dimension, including cases where the acceleration is dependent on time

M1.4.11 solve problems involving friction and problems on inclined planes

Both rough and smooth planes are required.

Mechanics 2 (AH)

M2.1 Motion in a horizontal circle with uniform angular velocity

M2.1.1 know the meaning of the terms angular velocity and angular acceleration

M2.1.2 know that for motion in a circle of radius r, the radial and tangential components of velocity are 0 and $r\dot{\theta}\mathbf{e}_\theta$ respectively, and of acceleration are $-r\dot{\theta}^2\mathbf{e}_r$ and $r\ddot{\theta}\mathbf{e}_\theta$ respectively, where $\mathbf{e}_r = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ and $\mathbf{e}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$ are the unit vectors in the radial and tangential directions, respectively

Vectors should be used to establish these, starting from $\mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$, where r is constant and θ is varying.

M2.1.3 know the particular case where $\theta = \omega t$, ω being constant, when the equations are $\mathbf{r} = r\cos(\omega t)\mathbf{i} + r\sin(\omega t)\mathbf{j}$;
$\mathbf{v} = -r\omega\sin(\omega t)\mathbf{i} + r\omega\cos(\omega t)\mathbf{j}$;
$\mathbf{a} = -r\omega^2\cos(\omega t)\mathbf{i} - r\omega^2\sin(\omega t)\mathbf{j}$;
from which
$v = r\omega = r\dot{\theta}$; $a = r\omega^2 = r\dot{\theta}^2 = v^2/r$ and $\mathbf{a} = -\omega^2\mathbf{r}$

M2.1.4 apply these equations to motion in a horizontal circle with uniform angular velocity including skidding and banking and other applications

Examples should include motion of cars round circular bends, with skidding and banking, the 'wall of death', the conical pendulum, etc.

M2.1.5 know Newton's inverse square law of gravitation, namely that the magnitude of the gravitational force of attraction between two particles is inversely proportional to the square of the distance between the two particles

M2.1.6 apply this to simplified examples of motion of satellites and moons

Circular orbits only.

M2.1.7 find the time for one orbit, height above surface, etc

M2.2 Simple harmonic motion

M2.2.1 know the definition of simple harmonic motion (SHM) and the meaning of the terms oscillation, centre of oscillation, period, amplitude, frequency

M2.2.2 know that SHM can be modelled by the equation $\ddot{x} = -\omega^2 x$

M2.2.3 know the solutions $x = a\sin(\omega t + \alpha)$ and the special cases $x = a\sin(\omega t)$ and $x = a\cos(\omega t)$, of the SHM equation

At this stage these solutions can be verified or established from $\mathbf{r} = a\cos(\omega t)\mathbf{i} + a\sin(\omega t)\mathbf{j}$ rotating round a circle. Solution of second order differential equations is not required.

M2.2.4 know and be able to verify that $v^2 = \omega^2(a^2 - x^2)$, where $v = \dot{x}$; $T = 2\pi/\omega$; maximum speed is ωa, the magnitude of the maximum acceleration is $\omega^2 a$ and when and where these arise

Proof using differential equations is not required here but will arise in the section of work on motion in a straight line later in this unit.

M2.2.5 know the meaning of the term tension in the context of elastic strings and springs

M2.2.6 know Hooke's law, the meaning of the terms natural length, l, modulus of elasticity, λ, and stiffness constant, k, and the connection between them, $\lambda = kl$

M2.2.7 know the equation of motion of an oscillating mass and the meaning of the term position of equilibrium

M2.2.8 apply the above to the solution of problems involving SHM

These will include problems involving elastic strings and springs, and small amplitude oscillations of a simple pendulum but not the compound pendulum.

M2.3 Principles of momentum and impulse

M2.3.1 know that force is the rate of change of momentum

This was introduced in Mechanics 1 (AH).

M2.3.2 know that impulse is change in momentum i.e. $\mathbf{I} = m\mathbf{v} - m\mathbf{u} = \int \mathbf{F}dt$

M2.3.3 understand the concept of conservation of linear momentum

M2.3.4 solve problems on linear motion such as motion in lifts, recoil of a gun, pile-drivers, etc.

The equation $\mathbf{F} = m\mathbf{a}$ is again involved here. Equations of motion with constant acceleration could recur.

M2.4 Principles of work, power and energy

M2.4.1 know the meaning of the terms work, power, potential energy, kinetic energy

M2.4.2 understand the concept of work

Candidates should appreciate that work can be done by or against a force.

M2.4.3 calculate the work done by a constant force in one and two dimensions, ie, $W = Fd$ (one dimension); $W = \mathbf{F}.\mathbf{d}$ (two dimensions)

M2.4.4 calculate the work done in rectilinear motion by a variable force using integration, i.e. $W = \int \mathbf{F}.\mathbf{i}\,dx$; $W = \int \mathbf{F}.\mathbf{v}\,dt$, where $\mathbf{v} = \frac{dx}{dt}\mathbf{i}$

M2.4.5 understand the concept of power as the rate of doing work, i.e. $P = \frac{dW}{dt} = \mathbf{F}.\mathbf{v}$ (constant force), and apply this in practical examples

Examples can be taken from transport, sport, fairgrounds, etc.

M2.4.6 understand the concept of energy and the difference between kinetic (E_K) and potential (E_P) energy

M2.4.7 know that $E_K = \frac{1}{2}mv^2$

M2.4.8 know that the potential energy associated with:

a. a uniform gravitational field is $E_P = mgh$

b. Hooke's law is $E_P = \frac{1}{2}k(\text{extension})^2$

Link with simple harmonic motion.

c. Newton's inverse square law is $E_P = \frac{GMm}{r}$

Link with motion in a horizontal circle.

M2.4.9 understand and apply the work-energy principle

M2.4.10 understand the meaning of conservative forces like gravity, and non-conservative forces like friction

M2.4.11 know and apply the energy equation $E_K + E_P =$ constant, including the situation of motion in a vertical circle

M2.5 Motion in a straight line, where the solution of first order differential equations is required

M2.5.1 know that $a = v\frac{dv}{dx}$ as well as $\frac{dv}{dt}$

M2.5.2 use Newton's law of motion, $\mathbf{F} = m\mathbf{a}$, to form first order differential equations to model practical problems, where the acceleration is dependent on displacement or velocity, i.e. $\frac{dv}{dt} = f(v)$; $v\frac{dv}{dx} = f(x)$; $v\frac{dv}{dx} = f(v)$.

M2.5.3 solve such differential equations by the method of separation of variables

It may be necessary to teach this solution technique, depending on the mathematical background of the candidates.

Examples will be straightforward with integrals which are covered in Mathematics 1, 2 (AH). If more complex, then the anti-derivative will be given.

M2.5.4 derive the equation $v^2 = \omega^2(a^2 - x^2)$ by solving $v\frac{dv}{dx} = -\omega^2 x$

M2.5.5 know the meaning of the terms terminal velocity, escape velocity and resistance per unit mass and solve problems involving differential equations and incorporating any of these terms or making use of $F = \frac{P}{v}$

This section can involve knowledge and skills from other

M1.1 MOTION IN A STRAIGHT LINE

Outcome Content

Know the meaning of position, displacement, velocity, uniform speed, uniform acceleration, scalar quantity, vector quantity.

Draw, interpret and use distance/time, velocity/time and acceleration/time graphs.

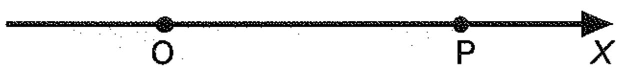
Know that the area under a velocity/time graph represents the distance travelled.

Know that the rates of change $v = \frac{ds}{dt} = \dot{x}$ and $a = \frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{d\dot{x}}{dt} = \dot{v} = \ddot{x}$

Solve problems involving motion in one dimension where the acceleration is dependent on time, i.e. $a = \frac{dv}{dt} = f(t)$.

You should be aware of the distinction between **scalar** and **vector** quantities.
Scalar quantities have magnitude (size) only whereas **vector** quantities have both magnitude and direction.

i.e.	**Scalars**	**Vectors**
	distance	displacement
	speed	velocity
	time	acceleration
	temperature	force



You should know that if a particle P is moving in a straight line along the X-axis and $\underline{i}$ is the unit vector in the positive direction of the X-axis then:

The **displacement** of P from O at any instant is the vector $\underline{x} = x\underline{i}$, where x = OP.
If P is to the right of O, x is positive and if P is to the left of O, x is negative.

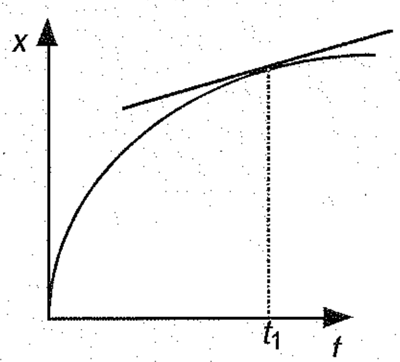
The **velocity** of the particle is the rate of change of its displacement and is the vector $\underline{v} = \frac{dx}{dt}\underline{i}$.

If P is moving to the right $\frac{dx}{dt}$ is positive

and if P is moving to the left $\frac{dx}{dt}$ is negative.

If $|\underline{v}|$, the speed, is denoted by v then $v = \frac{dx}{dt}$ and $x = \int v\,dt$.

$\frac{dx}{dt}$ is sometimes denoted by $\dot{x}$.



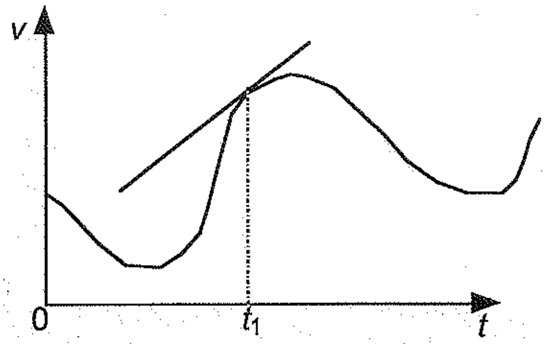
Gradient of graph at $t = t_1$ = $\frac{dx}{dt}$ = velocity at $t = t_1$

The **acceleration** of the particle is the rate of change of its velocity and is the vector $\underline{a} = \frac{dv}{dt}\underline{i}$.

If the velocity is increasing $\frac{dv}{dt}$ is positive and if the velocity is decreasing $\frac{dv}{dt}$ is negative.

If $|\underline{a}|$ is denoted by a then $a = \frac{dv}{dt}$ and $v = \int a\,dt$.

Also $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$ and $\frac{d^2x}{dt^2}$ is sometimes denoted by $\ddot{x}$.



Gradient of graph at $t = t_1$ $= \frac{dv}{dt} =$ acceleration at $t = t_1$

Area under graph $t = 0$ to $t = t_1$ $= \int_0^{t_1} v\,dt =$ displacement at $t = t_1$

The area under an acceleration / time graph from $t = 0$ to $t = t_1$ is

$$\int_0^{t_1} a\,dt = \text{ velocity at } t = t_1.$$

WORKED EXAMPLES

Example 1

A body moves along the X-axis with velocity, measured in ms-1, given by

$$\underline{v} = (3t^2 - 18t + 15)\underline{i}$$

where $\underline{i}$ is the unit vector in the positive direction of the X-axis and t is the time in seconds from the start of the motion.

At the start of the motion the displacement of the body from the origin is 30m. Find:

a) the initial speed of the body;
b) the values of t for which the body is at rest;
c) the acceleration of the body when $t = 6$;
d) the displacement of the body from O when $t = 3$.

Solution

a) $v = 3t^2 - 18t + 15$

When $t = 0$

$v = 3 \times 0^2 - 18 \times 0 + 15$

$v = 15 \text{ ms}^{-1}$

b) Body at rest so $v = 0$

$3t^2 - 18t + 15 = 0$

$3(t^2 - 6t + 5) = 0$

$3(t-1)(t-5) = 0$

So at rest when $t = 1$ and 5 s

c) Using $a = \frac{dv}{dt}$ then $a = 6t - 18$

When $t = 6$

$a = 6 \times 6 - 18$

$a = 18 \text{ ms}^{-2}$

d) Using $v = \frac{dx}{dt}$ and rearranging and integrating gives

$x = \int v\,dt = t^3 - 9t^2 + 15t + c$ where c is the constant of integration

When $t = 0$ $x = 30$ so $c = 30$ and $x = t^3 - 9t^2 + 15t + 30$

When $t = 3$ $x = 3^3 - 9 \times 3^2 + 15 \times 3 + 30 = 21$ m

Displacement from O is 21 m in the positive direction.

Example 2

A body moves along the X-axis from rest at the origin with acceleration, measured in ms^{-2}, given by

$$\underline{a} = (2 - \sqrt{t})\underline{i}$$

where $\underline{i}$ is the unit vector in the positive direction of the X-axis and t is the time in seconds from the start of the motion.

a) Show that the speed increases to a maximum value then decreases. Find this value.

b) Find

i) the time till the body is instantaneously at rest again

ii) the time before it again passes through its starting point.

Solution

a) If a body is accelerating its speed will increase.

For $a = 2 - \sqrt{t}$ $a = 0$ when $2 - \sqrt{t} = 0$

So speed is increasing until $t = 4$.

After $t = 4$ $a < 0$ so body starts to slow down.

There must be a stationary value at $t = 4$ and it is a maximum.

This could be confirmed by using a gradient table.

Using $\frac{dv}{dt} = a$ and rearranging and integrating

$v = \int a\,dt = \int (2 - \sqrt{t})dt = 2t - \frac{2}{3}t^{\frac{3}{2}} + c$ where c is the constant of integration

When $t = 0$, $v = 0$ so $c = 0$

$v = 2t - \frac{2}{3}t^{\frac{3}{2}}$ and when $t = 4$ $v = 2 \times 4 - \frac{2}{2} \times 4^{\frac{3}{2}} = \frac{8}{3}$ ms^{-1}

b)

i) Body instantaneously at rest so $v = 0$, so $2t - \frac{2}{3}t^{\frac{3}{2}} = 0$ which gives $t = 0$ or $t = 9$ and so it is at rest again after 9 seconds.

ii) At starting point $x = 0$

Using $\frac{dx}{dt} = v$ and rearranging and integrating

$x = \int v\,dt = \int (2t - \frac{2}{3}t^{\frac{3}{2}})dt = t^2 - \frac{4}{15}t^{\frac{5}{2}} + c$

When $t = 0$, $x = 0$ so $c = 0$

$x = t^2 - \frac{4}{15}t^{\frac{5}{2}}$

When $x = 0$, $t^2 - \frac{4}{15}t^{\frac{5}{2}} = 0$

$$t^2(1 - \frac{4}{15}t^{\frac{1}{2}}) = 0$$

$$t = 0 \text{ or } t^{\frac{1}{2}} = \frac{15}{4}$$

$$t = 0 \text{ or } t = \frac{225}{16} \text{ s}$$

The body passes through the starting point again after about $14 \cdot 06$ seconds.

Example 3

A particle starts from rest at the origin and moves along the *X*-axis with acceleration, measured in ms^{-2}, given by $\underline{a} = (6 - 2t)\underline{i}$, where $\underline{i}$ is the unit vector in the positive direction of the *X*-axis and *t* is the time in seconds from the start of the motion.

Find the maximum speed of the particle.
Find also the displacement of the particle on reaching the maximum speed.

Solution

Using $\frac{dv}{dt} = a$ and rearranging and integrating

$v = \int a\,dt = \int (6 - 2t)\,dt = 6t - t^2 + c$

When $t = 0,\ v = 0$ so $c = 0$

$v = 6t - t^2$

Maximum speed occurs when $a = 0$ so $6 - 2t = 0$ gives $t = 3$

Max speed $= 6 \times 3 - 3^2 = 9\ ms^{-1}$

Using $\frac{dx}{dt} = v$ then $x = \int v\,dt = \int (6t - t^2)\,dt = 3t^2 - \frac{1}{3}t^3 + c$

Starts from rest so when $t = 0,\ x = 0$ so $c = 0$

$x = 3t^2 - \frac{1}{3}t^3$

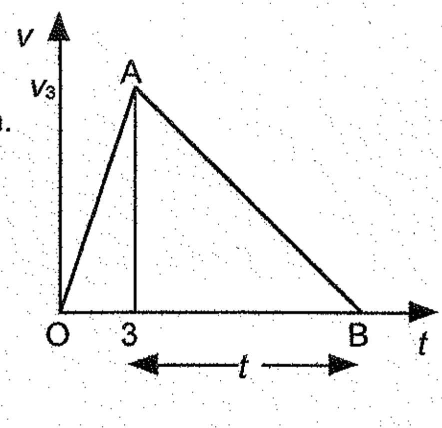
At max speed $t = 3$ so $x = 3 \times 3^2 - \frac{1}{3} \times 3^3 = 18$ m

Example 4

A particle starts from rest and moves in a straight line with uniform acceleration $6\,ms^{-2}$ for 3 seconds. It is then brought to rest with uniform acceleration of $-2\,ms^{-2}$.
Draw a velocity/time graph and use it to find the distance travelled by the particle.

Solution

Acceleration is the rate of change of velocity and **uniform** acceleration means **constant** acceleration. This rate of change will be the gradient in the corresponding velocity/time graph.



Gradient OA = acceleration in first 3 seconds

$m_{OA} = \frac{v_3}{3} = 6$ so the maximum speed $v_3 = 18\ ms^{-1}$

The distance travelled is the area under the velocity / time graph.

So $\frac{v_3}{t} = 2$ where *t* is the time to come back to rest.

Which gives $\frac{18}{t} = 2$ so $t = 9$ s

Area $\Delta OAB = \frac{1}{2} \times 12 \times 18 = 108$

Distance travelled = 108 m

Exercise M1.1-1

In all of the questions motion is in a straight line, $\underline{i}$ is the unit vector in the positive direction of the X-axis and x, v, a and t have their normal meanings.

1. If $\underline{x} = (t^3 + t)\underline{i}$ find v when $t = 3$ s.
2. If $\underline{x} = (5t^2 - t^3)\underline{i}$ find a when $t = 1$ s.
3. If $\underline{v} = t^3\underline{i}$ find a when $t = 2$ s.
4. If $\underline{v} = (4t + 5)\underline{i}$ and $x = 10$ m when $t = 1$ s, find x when $t = 2$ s.
5. If $\underline{a} = 6t\underline{i}$ and the body is initially at rest, find v when $t = 4$ s.
6. If $\underline{a} = \frac{3}{4}t\underline{i}$ find x when $t = 2$ s given that $v = 4\,\text{ms}^{-1}$, and $x = 10$ m when $t = 4$ s.
7. If $\underline{x} = (t^2 - 3)\underline{i}$ find:
   a) an expression for the velocity of the body at time t
   b) the value of t when the speed is $8\,\text{ms}^{-1}$
   c) the displacement of the body from O when $v = 8\,\text{ms}^{-1}$.
8. If $\underline{x} = (2t^3 - 21t^2 + 60t)\underline{i}$ find:
   a) the values of t when the body is at rest
   b) the initial velocity of the body
   c) an expression for the acceleration of the body at time t
   d) the initial acceleration of the body.
9. If $\underline{v} = (8t - 3t^2)\underline{i}$ and the body is initially at O, find:
   a) an expression for the acceleration of the body at time t
   b) an expression for the displacement of the body from O at time t
   c) how far the body is from O when $t = 3$ s.
10. If $\underline{v} = (3t - 2)(t - 4)\underline{i}$ and $x = 8$ m when $t = 1$ s, find:
    a) the initial velocity of the body
    b) the values of t when the body is at rest
    c) the acceleration of the body when $t = 3$ s
    d) the distance the body is from O when $t = 2$ s.
11. If $\underline{a} = 2t\underline{i}$ and initially the body is at rest at O, find the velocity of the body when $t = 3$ s and the distance the body is then from O.
12. A stone is dropped from a point O at the top of a cliff of height 80 m. The distance x m of the stone below O after t s is given by $x = 5t^2$.
    a) find the velocity of the stone just before it hits the ground
    b) show that the acceleration of the stone is constant.

13. A stone is thrown vertically upwards from a point O. The height x m of the stone above O is given by $x = 20t - 5t^2$.

a) What is the greatest height above O reached by the stone and how long does it take to reach this height?

b) What is the initial speed of the stone and what its speed on returning to O?

14. The displacement of a particle from O is given by $\underline{x} = (bt^2 + ct)\underline{i}$, where b and c are constants.

a) What is the initial displacement of P?

b) Find an expression for the speed v and determine the initial velocity of P.

c) Show that a is constant.

d) By eliminating t from your equations for x and v, show that $v^2 = c^2 + 4bx,\ \ b \neq 0$.

15. A man runs from his house O to the local shop A along a straight road. He realises that he has forgotten to bring any money, so turns round and runs back to O again. During his journey his distance x m from O is given by

$$x = \frac{1}{80}(30t^2 - t^3)$$

a) Find the distance from O to A.

b) Find the total time for his journey.

c) Find his maximum speed during the journey.

16. A body moves along a straight line with acceleration given by $a = \frac{7t}{36}$ where t is the time in seconds. When $t = 0$ the body is at rest at an origin O. The acceleration continues until $t = 6$, when it is then decelerated to rest. During this deceleration $a = -\frac{t}{4}$. Find the value of t when the body comes to rest and the displacement of the body from O at that time.

Answers

1. 28 ms^{-1}

2. 4 ms^{-2}

3. 12 ms^{-2}

4. 21 m

5. 48 ms^{-1}

6. 7 m

7. a) $v = 2t$
b) 4 s
c) 13 m

8. a) 2 and 5 s
b) $60\underline{i}$
c) $a = 12t - 42$
d) $-42\underline{i}$

9. a) $\underline{a} = (8 - 6t)\underline{i}$
b) $\underline{x} = (4t^2 - t^3)\underline{i}$
c) 9 m

10. a) $8\underline{i}$
b) 2/3 and 4 s
c) $4\underline{i}$
d) 2 m

11. a) $\underline{v} = 9\underline{i}$
b) 9 m

12. a) $\underline{v} = 40\underline{i}$
b) $a = 10$ a constant

13. a) 20 m
b) 20 ms^{-1}

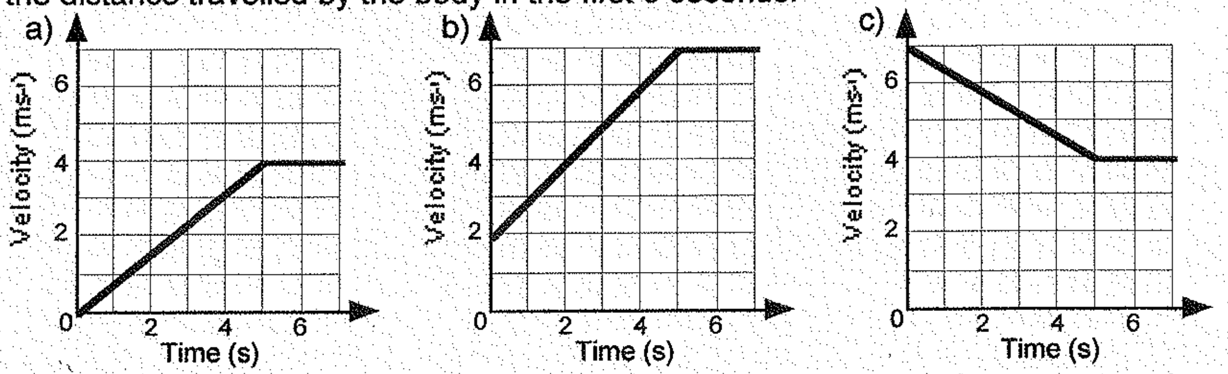
14. a) $\underline{0}$
b) $c\underline{i}$
c) $a = 2b$ which is a constant.

15. a) 50 m
b) 30 s
c) 15/4 ms^{-1}

16. 8 s
$32/3\,\underline{i}$

Exercise M1.1-2

1. Each of the following velocity/time graphs are for a body which accelerates uniformly for a time period of 5 seconds after which it maintains its final velocity. In each case find:
   i) the acceleration of the body during the 5 seconds
   ii) the distance travelled by the body in the first 6 seconds.



2. A cyclist rides along a straight road from point A to point B. He starts from rest, accelerates uniformly to reach a speed of $12\,ms^{-1}$ in 8 seconds. He maintains this speed for a further 20 seconds and then uniformly slows down to rest at B.
   If the whole journey lasts 34 seconds, draw a velocity/time graph and from it find:
   a) his acceleration for the first part of the motion
   b) his acceleration (deceleration) for the last part of the journey
   c) the total distance travelled.

3. A particle is initially at rest at a point A on a straight line ABCD. The particle moves from A to B with uniform acceleration, reaching B with a speed of $12\,ms^{-1}$ after 2 seconds. The acceleration then alters to a constant $1\,ms^{-2}$ and 8 seconds after leaving B the particle reaches C. The particle then slows down uniformly to come to rest at D after a further 10 seconds.
   Draw a velocity/time graph for the motion and from it find:
   a) the acceleration of the particle when travelling from A to B
   b) the speed of the particle on reaching C
   c) the acceleration of the particle when travelling from C to D
   d) the total distance from A to D.

4. Two stations A and B are a distance of $6x$ m apart along a straight track. A train starts from rest at A and accelerates uniformly to a speed $v\,ms^{-1}$, covering a distance of x m. The train then maintains this speed until it has travelled a further $3x$ m, it then slows down at a uniform rate to rest at B. Make a sketch of the velocity/time graph for the motion and show that if T is the time taken for the train to travel from A to B then $T = \dfrac{9x}{v}$ seconds.

5. An elevator travels from rest at the ground floor of a building to rest at the top floor, 50 m above, taking 18 seconds. It accelerates uniformly for 10 m, travels with constant speed for 32 m and then decelerates. Find:
   a) the constant speed
   b) the acceleration and the deceleration.

6. A particle moves in a straight line from rest at A to rest at B. It accelerates uniformly at a ms^{-2}, moves with constant speed Vms^{-1} and then decelerates uniformly at $\frac{1}{2}a$ ms^{-2}. If the total time for the journey is Ts and the distance AB is $\frac{5}{32}aT^2$ m, show that $V = \frac{1}{4}aT$ ms^{-1}.

7. A particle moves so that $x = 6t^2 - t^3$.
   a) Show that the particle is initially at O and returns to O after 6 seconds.
   b) Find the maximum displacement of the particle from O in the time interval $0 \le t \le 6$.
   c) Sketch a graph of the speed of the particle against t. Find the maximum speed of the particle in the interval $0 \le t \le 6$.

Answers

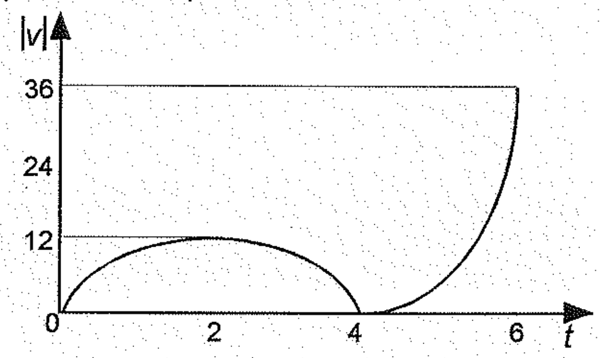
1. a) i) 0·8 ms^{-2} ii) 14 m
b) i) 1 ms^{-2} ii) 29·5 m
c) i) -0·6 ms^{-2} ii) 31·5 m

2. a) 1·5 ms^{-2}
b) -2 ms^{-2}
c) 324 m

3. a) 6 ms^{-2}
b) 20 ms^{-1}
c) -2 ms^{-2}
d) 240 m

5. a) $3\cdot\dot{7}$ ms^{-1}
b) 0·71 and 0·89 ms^{-2}

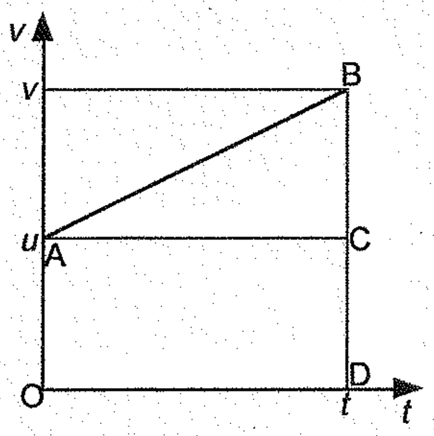
7. b) 32 m
c) Maximum speed is 36 ms^{-1}



Equations of Motion of a body moving in a straight line with line with uniform acceleration - Graphical Approach

The method of obtaining the equations given in this section is for information only, you are required to use calculus methods to derive the equations in this course.

Consider a particle moving in a straight line with constant acceleration so that $\underline{a} = a\underline{i}$ with initial velocity $\underline{u} = u\underline{i}$ and final velocity $\underline{v} = v\underline{i}$ after time t.
The velocity/time graph for such a situation is shown below.



a = rate of change of velocity
= gradient of AB

$$= \frac{BC}{AC}$$

$$\text{so } a = \frac{v-u}{t}$$

Rearranging gives $v = u + at$ [1]

Using distance travelled = area under velocity / time graph

$$x = \text{area}OACD + \text{area}ABC$$

$$x = ut + \tfrac{1}{2}t(v-u)$$

but $at = v - u$ from [1]

$$\text{so } x = ut + \tfrac{1}{2}at^2 \qquad [2]$$

It is common in Physics/Mechanics to use s for the distance travelled so the last result is normally seen as

$$s = ut + \tfrac{1}{2}at^2$$

Equations [1] and [2] form the basis for straight line motion uniform acceleration calculations.

Outcome Content

Derive, by calculus methods, and use the equations governing motion in a straight line with constant acceleration, namely:

$v = u + at$, $s = ut + \frac{1}{2}at^2$ and from these

$v^2 = u^2 + 2as$, $s = \dfrac{(u+v)t}{2}$

Solve analytically problems involving motion in one dimension under constant acceleration, including vertical motion under constant gravity.

It has to be clearly emphasised that these equations are **only for motion with constant acceleration**. The general method is to use calculus techniques as covered earlier.

Proof using calculus

Rate of change of velocity = acceleration

so using $\dfrac{dv}{dt} = a$ and rearranging and integrating

$v = \int a\,dt$ but since a is constant

$v = a\int dt$

so $v = at + c$ where c is a constant of integration

When $t = 0$ $v = u$

so $c = u$

which gives $v = u + at$ [1]

Using $\dfrac{dx}{dt} = v$ and rearranging and integrating

$x = \int v\,dt = \int (u + at)dt$

so $x = ut + \frac{1}{2}at^2 + c_1$ where c_1 is a constant of integration

When $t = 0$ $x = 0$ so $c_1 = 0$

$x = ut + \frac{1}{2}at^2$ [2]

These are the two equations obtained previously using the graphical approach. Another two equations can be obtained by combining [1] and [2] as follows:

From [1] $t = \dfrac{v-u}{a}$, substituting for t in [2] and replacing x by s

$$s = \frac{u(v-u)}{a} + \frac{1}{2}\frac{(v-u)^2}{a}$$

so $2as = 2u(v-u) + (v-u)^2$

$2as = 2uv - 2u^2 + v^2 - 2uv + u^2$

$2as = v^2 - u^2$

Giving $v^2 = u^2 + 2as$ [3]

Rearranging [3] gives $s = \dfrac{v^2 - u^2}{2a}$ and using the difference of squares

$$s = \frac{(v+u)(v-u)}{2a}$$

From [1] $t = \dfrac{v-u}{a}$ so substituting for $\dfrac{v-u}{a}$ in [3] gives

$s = \dfrac{(u+v)t}{2}$ [4]

WORKED EXAMPLES

Example 1
A particle moving in a straight line with constant acceleration increases its velocity from $4\,ms^{-1}$ to $16\,ms^{-1}$ in 6 seconds. Find the constant acceleration and the distance travelled during the 6 seconds.

Solution

$u = 4\,ms^{-1}$ $\quad v = 16\,ms^{-1}$ $\quad t = 6\,s$

For a use $v = u + at$

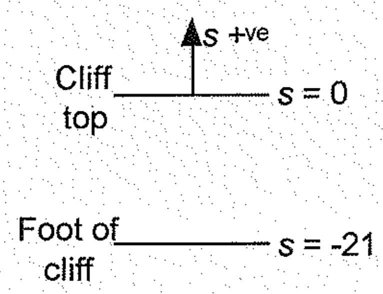
$$16 = 4 + 6a$$
$$6a = 12$$
$$a = 2\ ms^{-2}$$

For s use $s = ut + \frac{1}{2}at^2$

$$s = 4 \times 6 + \tfrac{1}{2} \times 2 \times 6^2$$
$$s = 60\ m$$

Example 2
A ball is thrown upwards, with a speed $7{\cdot}7\,ms^{-1}$, from the top of a sheer cliff 21 m high in such a way that on the downwards part of the motion the edge of the cliff is just missed and the ball continues downwards. Find:
a) the time taken for the ball to reach the foot of the cliff
b) the velocity of the ball at the instant it hits the ground.

Solution



a) Taking upwards as the positive direction

$u = 7{\cdot}7\ ms^{-1}$ and $a = -9{\cdot}8\ ms^{-2}$

Using $s = ut + \frac{1}{2}at^2$

$$-21 = 7{\cdot}7t - 4{\cdot}9t^2$$

so $49t^2 - 77t - 210 = 0$

$$7t^2 - 11t - 30 = 0$$
$$(7t + 10)(t - 3) = 0$$

$t = -\dfrac{10}{7}$ or $t = 3$ but $t > 0$ so $t = 3$

Time to reach foot of cliff is 3 s.

b) Using $v = u + at$

$$v = 7{\cdot}7 - 9{\cdot}8 \times 3$$
$$v = -21{\cdot}7$$

so the velocity is $21{\cdot}7\ ms^{-1}$ vertically downwards.

Exercise M1.1-3

1. In travelling the 70 cm along a rifle barrel, a bullet uniformly accelerates from its initial state of rest to an exit velocity of 210 ms^{-1}. Find the acceleration involved and the time for which the bullet is in the barrel.

2. According to the highway code, a car travelling at 20 ms^{-1} requires a minimum braking distance of 30 m. What deceleration is this and what will the length of time be to stop?

3. A particle is projected away from an origin O with an initial velocity 0·25 ms^{-1}. The particle travels in a straight line and accelerates at 1·5 ms^{-2}.
   Find a) how far the particle is from O after 3 seconds
   b) the distance travelled by the particle during the fourth second after projection.

4. A particle travels in a straight line with uniform acceleration. The particle passes through three points A, B and C lying in that order on the line, at times $t = 0$, $t = 2$ s and $t = 5$ s respectively. If BC = 30 m and the speed of the particle when at B is 7 ms^{-1} find the acceleration of the particle and its speed when at A.

5. A, B and C are three points which lie in order on a straight road with AB = 95 m and BC = 80 m. A car is travelling along the road in the direction ABC with constant acceleration a ms^{-2}. The car passes through A with speed u ms^{-1}, reaches B five seconds later and C two seconds after that. Find the values of u and a.

6. A car A, travelling at a constant velocity of 25 ms^{-1}, passes a stationary car B. Two seconds later car B sets off to follow A, accelerating at a uniform 6 ms^{-2}. How far does B travel before catching up with A?

In the following questions take the magnitude of the acceleration due to gravity as 9·8 ms^{-2}

7. A ball is thrown vertically upwards with an initial speed of 14 ms^{-1}. Find the height above ground level of the highest point reached and the total time for which the ball was in the air.

8. A stone is thrown vertically upwards with a speed of 20 ms^{-1} from a point at a height h metres above ground level. If the stone hits the ground 5 seconds later, find h.

9. A stone is projected vertically upwards from ground level at a speed of 24·5 ms^{-1}. Find how long after projection the stone is at a height of 19·6 m above the ground:
   (i) for the first time (ii) for the second time.
   For how long is the stone at least 19·6 m above ground level?

10. A ball is held 1·6 m above a concrete floor and released. The ball hits the floor and rebounds with half the speed it had prior to impact. Find the greatest height the ball reaches after:
    a) the first bounce b) the second bounce.

11. A bullet is fired vertically upwards at a speed of 147 ms^{-1}. Find the length of time for which the bullet is at least 980 m above the level of projection.

12. Two stones are thrown from the same point at the same time, one vertically upwards with speed 30 ms^{-1}, and the other vertically downwards at 30 ms^{-1}. Find how far apart the stones are after 3 seconds.

Answers

1. 31 500 ms^{-2} 1/150 s	**2.** -20/3 ms^{-2} 3 s	**3.** 7·5 m 5·5 m	**4.** 2 ms^{-2} 3 ms^{-1}
5. 4 ms^{-1} 6 ms^{-2}	**6.** 300 m	**7.** 10 m 20/7 s	**8.** 22·5 m
9. (i) 1 s (ii) 4 s 3 s	**10.** a) 0·4 m b) 0·1 m	**11.** 10 s	**12.** 180 m

M1.2 POSITION, VELOCITY AND ACCELERATION VECTORS INCLUDING RELATIVE MOTION

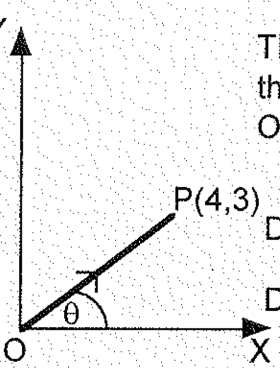
Outcome Content
Know the meaning of the terms relative position, relative velocity and relative acceleration, air speed, ground speed and nearest approach.
Be familiar with the notation $\underline{r}_P$ for the position vector of P $\underline{v}_P = \dot{\underline{r}}_P$ for the velocity vector of P $\underline{a}_P = \dot{\underline{v}}_P = \ddot{\underline{r}}_P$ for the acceleration of P $\vec{PQ} = \underline{r}_Q - \underline{r}_P$ for the position vector of Q relative to P $\underline{v}_Q - \underline{v}_P = \dot{\underline{r}}_Q - \dot{\underline{r}}_P$ for the velocity of Q relative to P $\underline{a}_Q - \underline{a}_P = \dot{\underline{v}}_Q - \dot{\underline{v}}_P = \ddot{\underline{r}}_Q - \ddot{\underline{r}}_P$ for the acceleration of Q relative to P
Resolve vectors into components in two and three dimensions.
Differentiate and integrate vector functions of time.
Use position, velocity and acceleration vectors and their components in two and three dimensions; these vectors may be functions of time.
Apply position, velocity and acceleration vectors to solve practical problems, including problems on navigation of ships and aircraft and on the effect of winds and currents.
Solve problems involving collisions and nearest approach.

Recap of Assumed Knowledge of Vectors
You should be aware of the following vector details which were introduced in the higher course:

1. Know the terms **position vector** and **unit vector**.

In the diagram the position of P is given by the coordinates (4,3). The position vector of P is drawn from the origin O and uses the coordinates of P as its components.

Position vector of P is $\vec{OP} = \underline{r}_P = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$



The position vector of P carries in the components the distance from O and the direction to P.

Distance OP $= \sqrt{4^2 + 3^2} = 5$ units

Direction θ, $\tan\theta = \dfrac{3}{4}$ so $\theta \approx 36 \cdot 9^0$

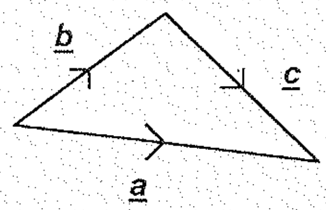
A unit vector has a magnitude of one unit. The main use of a unit vector is to define a direction. A unit vector for a desired direction can be obtained by dividing the components of a vector going in the required direction by its magnitude.

If $\underline{a} = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$ then the magnitude of $\underline{a}$, $|\underline{a}| = 5$ so a unit vector

in the same direction as $\underline{a}$ is $\hat{\underline{a}} = \frac{1}{5}\begin{pmatrix} 4 \\ 3 \end{pmatrix}$.

2. Know the properties of **vector addition** and **multiplication of a vector by a scalar**.
Vector addition can be done arithmetically or it can be done pictorially.

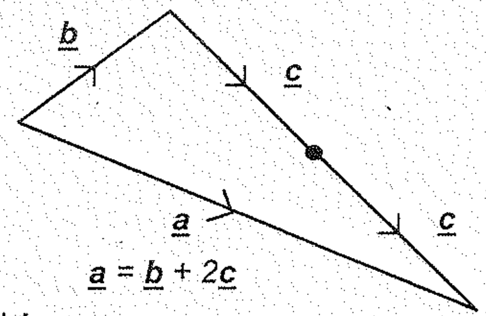
eg $\begin{pmatrix}4\\3\\1\end{pmatrix}+\begin{pmatrix}-3\\-6\\0\end{pmatrix}=\begin{pmatrix}1\\-3\\1\end{pmatrix}$ or



$\underline{a} = \underline{b} + \underline{c}$

Scalar multiplication of a vector can also be illustrated in these ways.

eg $\begin{pmatrix}4\\3\\1\end{pmatrix}+2\begin{pmatrix}-3\\-6\\0\end{pmatrix}=\begin{pmatrix}4\\3\\1\end{pmatrix}+\begin{pmatrix}-6\\-12\\0\end{pmatrix}=\begin{pmatrix}-2\\-9\\1\end{pmatrix}$ or



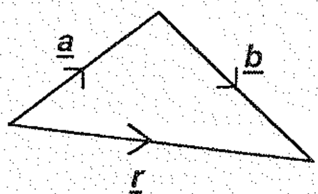
$\underline{a} = \underline{b} + 2\underline{c}$

3. Know and can apply the **basis vectors** $\underline{i}$, $\underline{j}$ and $\underline{k}$.
$\underline{i}$, $\underline{j}$ and $\underline{k}$ are unit vectors with directions corresponding to the X, Y and Z axes respectively.
The vector, in component form, $\begin{pmatrix}4\\3\\1\end{pmatrix}=4\underline{i}+3\underline{j}+\underline{k}$.

Resolving Vectors

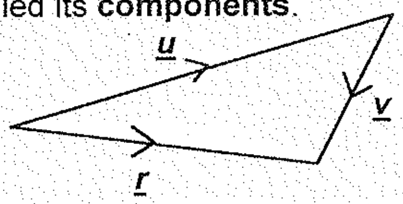
This is the opposite process to adding two or more vectors to obtain their sum. It provides the basis of much of the vector work in the latter part of this unit.

When **resolving a vector** we start with a single vector and replace it with two vectors.



$\underline{a} + \underline{b} = \underline{r}$ expresses the sum of the vectors $\underline{a}$ and $\underline{b}$ as a single vector $\underline{r}$ called the **resultant**.
$\underline{r} = \underline{a} + \underline{b}$ expresses $\underline{r}$ as the sum of two vectors $\underline{a}$ and $\underline{b}$ called its **components**.

$\underline{r}$ can be resolved into components in infinitely many ways. For example, in the diagram on the right the same vector $\underline{r}$ has now been resolved so that $\underline{r} = \underline{u} + \underline{v}$.



It is normal practice, in work in two dimensions to resolve a vector into two vectors at right angles to one another.

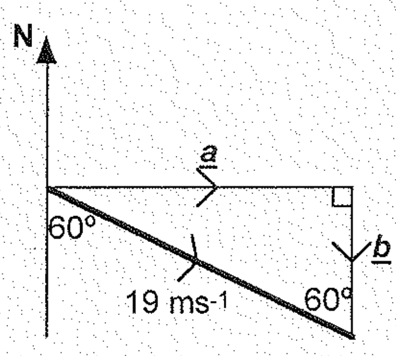
WORKED EXAMPLE

Example 1

The resultant of two velocities is a velocity of 19 ms-1, S 60° E. If one of the velocities is 10 ms-1 due east, find the magnitude and direction of the other velocity.

Solution

For the resultant 19 ms-1, S 60° E as shown in the diagram resolving into $\underline{i}$ and $\underline{j}$ component form.



$\underline{a} = 19\sin 60^0 \underline{i}$ and $\underline{b} = -19\cos 60^0 \underline{j}$

So resultant $= \dfrac{19\sqrt{3}}{2}\underline{i} - \dfrac{19}{2}\underline{j}$

Velocity 10 ms^{-1} due east in component form is $10\underline{i}$.

Let the components of the other velocity, say $\underline{u}$, be c and d.

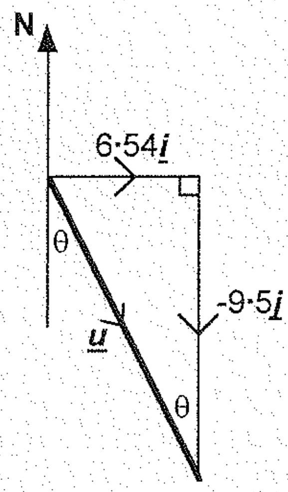
Then $c\underline{i} + d\underline{j} + 10\underline{i} = \dfrac{19\sqrt{3}}{2}\underline{i} - \dfrac{19}{2}\underline{j}$

So $c + 10 = \dfrac{19\sqrt{3}}{2}$ giving $c \approx 6\cdot 54....$

and $d = -\dfrac{19}{2} = -9\cdot 5$

The other vector, $\underline{u} = 6\cdot 54\underline{i} - 9\cdot 5\underline{j}$

For its magnitude and direction refer to the sketch opposite.



$|\underline{u}|^2 = 6\cdot 54^2 + 9\cdot 5^2$

$|\underline{u}|^2 = 131\cdot 9....$

$|\underline{u}| = 11\cdot 5$ ms^{-1}

For the direction

$\tan\theta = \dfrac{6\cdot 54}{9\cdot 5} = 0\cdot 679...$

$\theta \approx 34\cdot 2^0$

Direction $145\cdot 8^0$ or S $34\cdot 2^0$ E.

Exercise M1.2-1

1. Find in vector form the resultant of each of the following sets of velocities.

   a) $(6\underline{i} + 2\underline{j})$ ms^{-1}, $(2\underline{i} + 3\underline{j})$ ms^{-1}, $(-\underline{i} + 4\underline{j})$ ms^{-1}
   b) $(2\underline{i} - 5\underline{j})$ ms^{-1}, $(3\underline{i} + 7\underline{j})$ ms^{-1}, $(-6\underline{i} - 8\underline{j})$ ms^{-1}

2. Find by calculation the magnitude and direction of the resultant of each of the following pairs of velocities.
   a) 24 ms^{-1} due north, 7 ms^{-1} due east
   b) 5 ms^{-1} due north, 7 ms^{-1} S 60° E
   c) 72 ms^{-1} N 65° E, 20 ms^{-1} SE

3. If the resultant of $(3\underline{i} + 4\underline{j})$ ms^{-1} and $(a\underline{i} + b\underline{j})$ ms^{-1} is $(7\underline{i} - \underline{j})$ ms^{-1}, find the values of a and b.

4. If the resultant of $(a\underline{i} + b\underline{j})$ ms^{-1} and $(b\underline{i} - a\underline{j})$ ms^{-1} is $(10\underline{i} - 4\underline{j})$ ms^{-1}, find the values of a and b.

5. The resultant of two velocities is a velocity of 10 ms^{-1}, N 30° W. If one of the velocities is 10 ms^{-1} due west, find the magnitude and direction of the other velocity.

6. The resultant of two velocities is a velocity of 6 ms^{-1} due east. If one of the velocities is 5 ms^{-1}, N 30° W, find the magnitude and direction of the other velocity.

7. Find the vector of the given magnitude in the given direction.

   a) 10; $3\underline{i} + 4\underline{j}$ b) 39; $12\underline{i} - 5\underline{j}$ c) 4; $-\sqrt{3}\underline{i} - \underline{j}$

Answers

1. a) $(7\underline{i} + 9\underline{j})$ ms^{-1}
   b) $(-\underline{i} - 6\underline{j})$ ms^{-1}

2. a) 25 ms^{-1} direction 016·3° or N 16·3° E
   b) $\sqrt{39}$ ms^{-1} direction 076·1° or N 76·1° E
   c) ≈81·0 ms^{-1} direction 078·4° or N 78·4° E

3. $a = 4$ $b = -5$
4. $a = 7$ $b = 3$
5. 10 ms^{-1} direction 030° or N 30° E
6. ≈ 9·54 ms^{-1} direction 117·0° or S 63·0° E
7. a) $6\underline{i} + 8\underline{j}$
   b) $36\underline{i} - 15\underline{j}$
   c) $-2\sqrt{3}\underline{i} - 2\underline{j}$

Differentiation and Integration of Vector Functions of Time

At Higher we considered the position vectors of fixed points. Here we deal with the position vectors of moving particles or bodies, and so the vector's components are functions of time.

Differentiation

If P is a particle then we denote the position vector of P at time t by $\underline{r}_P$.

If $\underline{r}_P = x\underline{i} + y\underline{j} + z\underline{k}$, where x, y and z are functions of time, then the velocity vector of P is:

$$\underline{v}_P = \frac{d\underline{r}_P}{dt} = \frac{dx}{dt}\underline{i} + \frac{dy}{dt}\underline{j} + \frac{dz}{dt}\underline{k}.$$

Using the 'dot' notation for the differentiation with respect to time this becomes:

$$\underline{v}_P = \dot{\underline{r}}_P = \dot{x}\underline{i} + \dot{y}\underline{j} + \dot{z}\underline{k}.$$

The acceleration vector of P is $\underline{a}_P = \dfrac{d\underline{v}_P}{dt} = \dot{\underline{v}}_P = \ddot{\underline{r}}_P = \ddot{x}\underline{i} + \ddot{y}\underline{j} + \ddot{z}\underline{k}$.

Integration

Given the particle's velocity (acceleration) vector we can integrate to find its position (velocity) vector.

For example:

Suppose the velocity vector of a particle, P, is given by $\underline{v}_P = 2t\underline{i} + \underline{j} + 6t\underline{k}$.

Integrating gives $\underline{r}_P = \int \underline{v}_P dt = \int (2t\underline{i} + \underline{j} + 6t\underline{k})dt = t^2\underline{i} + t\underline{j} + 3t^2\underline{k} + \underline{c}$ where $\underline{c}$ is the constant vector of integration.

In practice we minimise the notation and simply write the result of the integration as $\underline{r}_P = t^2\underline{i} + t\underline{j} + 3t^2\underline{k} + \underline{c}$ and determine $\underline{c}$ from the given boundary conditions.

For example, suppose that the particle started from the point with position vector $\underline{r} = 2\underline{i} - \underline{j} + 5\underline{k}$.

Then $\underline{r}_P = 2\underline{i} - \underline{j} + 5\underline{k}$ when $t = 0$, so $\underline{c} = 2\underline{i} - \underline{j} + 5\underline{k}$ and $\underline{r}_P = (t^2+2)\underline{i} + (t-1)\underline{j} + (3t^2+5)\underline{k}$.

WORKED EXAMPLES

Example 2

A particle, P, moves from the point with position vector $-24\underline{i} - 72\underline{j} + 6\underline{k}$, with initial velocity $4\underline{i} + 5\underline{k}$, and subject to an acceleration of $4\underline{j} - 2\underline{k}$, where $\underline{i}$, $\underline{j}$ and $\underline{k}$ are unit vectors in the direction of the orthogonal axes Ox, Oy and Oz respectively, and t is the time, in seconds from the start of the motion. Distances are measured in metres and velocities in ms^{-1}.

Find an expression for the velocity of the particle at time t and calculate the speed of the particle when $t = 2$. Find the position vector of the particle at time t. When will the particle pass through the origin?

Solution

It is important to realise that the position and velocity vectors given are for a particular point in time, namely when $t = 0$. The acceleration vector given is constant for all points in time and is therefore the starting point in the problem's solution.

$\underline{a}_P = 4\underline{j} - 2\underline{k}$

Integrating gives $\underline{v}_P = 4t\underline{j} - 2t\underline{k} + \underline{c}_1$ $\quad$ $\underline{c}_1$ is used as another vector constant will be met later in the solution.

Now $\underline{v}_P = 4\underline{i} + 5\underline{k}$ when $t = 0$ so $\underline{c}_1 = 4\underline{i} + 5\underline{k}$
and $\underline{v}_P = 4\underline{i} + 4t\,\underline{j} + (5 - 2t)\underline{k}$.
When $t = 2$, $\underline{v}_P = 4\underline{i} + (4 \times 2)\underline{j} + (5 - 2 \times 2)\underline{k} = 4\underline{i} + 8\underline{j} + \underline{k}$.
Speed $= |\underline{v}_P| = \sqrt{4^2 + 8^2 + 1^2} = \sqrt{81} = 9\ \text{ms}^{-1}$.

Using $\underline{v}_P = 4\underline{i} + 4t\,\underline{j} + (5 - 2t)\underline{k}$.
Integrating gives $\underline{r}_P = 4t\underline{i} + 2t^2\underline{j} + (5t - t^2)\underline{k} + \underline{c}_2$
Now $\underline{r}_P = -24\underline{i} - 72\underline{j} + 6\underline{k}$ when $t = 0$ so $\underline{c}_2 = -24\underline{i} - 72\underline{j} + 6\underline{k}$
and $\underline{r}_P = (4t - 24)\underline{i} + (2t^2 - 72)\underline{j} + (6 + 5t - t^2)\underline{k}$

$$= 4(t - 6)\underline{i} + 2(t - 6)(t + 6)\underline{j} + (6 - t)(1 + t)\underline{k}$$

$$= (t - 6)\left[4\underline{i} + 2(t + 6)\underline{j} - (1 + t)\underline{k}\right]$$

Particle passes through the origin when $\underline{r}_P = \underline{0}$.
This occurs when $(t - 6) = 0$ so passes through the origin after 6 seconds.

Example 3

A particle, P, moves so that $\underline{r}_P = 4t\underline{i} + \dfrac{t^3}{4}\underline{j}$ for $0 \le t \le 6$, where $\underline{i}$ and $\underline{j}$ are perpendicular unit vectors and t is the time in seconds from the start of the motion.

Find the speed of the particle at the start and also when $t = 4$ (distances are measured in metres).

Find the acceleration of the particle at time t.

Solution

$$\underline{r}_P = 4t\underline{i} + \frac{t^3}{4}\underline{j}$$

Differentiating gives $\underline{v}_P = 4\underline{i} + \dfrac{3t^2}{4}\underline{j}$

When $t = 0$, $\underline{v}_P = 4\underline{i}$ so initial speed $= 4\ \text{ms}^{-1}$.

When $t = 4$, $\underline{v}_P = 4\underline{i} + \dfrac{3 \times 4^2}{4}\underline{j} = 4\underline{i} + 12\underline{j}$ so speed $= |\underline{v}_P| = \sqrt{4^2 + 12^2} = \sqrt{160} \approx 12{\cdot}6\ \text{ms}^{-1}$.

Using $\underline{v}_P = 4\underline{i} + \dfrac{3t^2}{4}\underline{j}$

Differentiating gives $\underline{a}_P = 0\underline{i} + \dfrac{3 \times 2t}{4}\underline{j} = \dfrac{3t}{2}\underline{j}$

Exercise M1.2-2

1. The position vector of a particle at time t is given by $\underline{r} = 2t\underline{i} + 3t^2\underline{j} - 4\underline{k}$.
   Find the velocity vector of the particle when $t = \frac{1}{2}$.
2. The velocity of a particle at time t is given by $\underline{v} = \frac{1}{3}\underline{i} - \frac{1}{4}t\underline{j}$. If the position vector of the particle at time $t = 0$ is $2\underline{k}$, find its position at $t = 4$.
3. A particle at rest at the origin at time $t = 0$, has velocity represented by the sum of the vectors

   $t^2\underline{i} + 3t\underline{j}$, $\quad -2t\underline{i} + 2\underline{j} - 3\underline{k}$, $\quad -4t\underline{j} + 3\underline{k}$

   Find the **resultant** velocity vector. Find also the instant the particle is stationary, and its position vector at that instant.
4. Find the velocity and acceleration of the particle with position vector $\underline{r} = 4t\underline{i} + 5t^2\underline{j} + 3\underline{k}$.
5. Find the position vector at time t of the particle which is projected from the origin with velocity $-3\underline{k}$ and acceleration $\underline{i} + t\underline{j}$.
6. A particle is launched at time $t = 0$ from position $-5\underline{i}$, with initial velocity $\underline{i} - 10\underline{j}$ and acceleration $4\underline{j} + (10 - 6t)\underline{k}$. Find the velocity and position vectors at time t, and find when the particle passes through the origin.
7. The position vector of a body is

   $$\underline{r} = 3t\underline{i} + (6t - \frac{3t^2}{2})\underline{j} + (2t - \frac{t^2}{2})\underline{k}$$

   Find the instant at which the body is moving parallel to the vector $\underline{i}$.
   Show that the acceleration of the body is constant, and find its magnitude.
8. The position vector of a particle is given by $\underline{r} = t^2\underline{i} + 2\underline{j}$. Find the velocity vector. Sketch the path of the particle and find the distance travelled in the time between $t = 2$ and $t = 5$.
9. The position vector of a particle P at time t is given by $\underline{r} = at^2\underline{i} + 2at\underline{j}$. Show that the path of P is a parabola.
   Find the speed of P at $t = 5$.
10. The position vectors of two particles A and B at time t are given by:

    $\underline{r}_A = t\underline{i} + (t^2 - 2t)\underline{j}$ and $\underline{r}_B = 2t\underline{i} + (4t^2 - 4t)\underline{j}$

    Show that A and B travel the same path.
    What is the difference between the motion of the two particles?

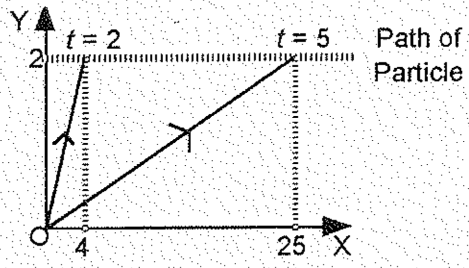
11. The position vector $\underline{r}$ of a particle at time t is given by $\underline{r} = 10t\underline{i} + (20t - 5t^2)\underline{j}$. Find the time at which the velocity of the particle is perpendicular to its initial velocity, and the position vector of the particle at this time.
12. The position vector $\underline{r}$ of a particle at time t is given by $\underline{r} = (t^2 + 1)\underline{i} + (t^3 - 2t)\underline{j}$. Find the position vector of the particle when its velocity and acceleration are perpendicular.

Answers

1. $\underline{v} = 2\underline{i} + 3\underline{j}$
2. $\underline{r} = \frac{4}{3}\underline{i} - 2\underline{j} + 2\underline{k}$
3. $\underline{v} = (t^2 - 2t)\underline{i} + (2 - t)\underline{j}$; $t = 2$ s;
4. $\underline{v} = 4\underline{i} + 10t\underline{j}$; $\underline{a} = 10\underline{j}$
5. $\underline{r} = \frac{1}{2}t^2\underline{i} + \frac{1}{6}t^3\underline{j} - 3t\underline{k}$ $\quad \underline{r} = -\frac{4}{3}\underline{i} + 2\underline{j}$
6. $\underline{v} = \underline{i} + (4t - 10)\underline{j} + (10t - 3t^2)\underline{k}$

   $\underline{r} = (t-5)\underline{i} + (2t^2 - 10t)\underline{j} + (5t^2 - t^3)\underline{k}$

   Passes through O when $t = 5$ s.
7. $t = 2$ and $|\underline{a}| = \sqrt{10}$ ms^{-2}
8. $\underline{v} = 2t\,\underline{i}$



9. $x = \dfrac{1}{4a}y^2$ which is the equation of a parabola with axis of symmetry the X axis.

   Speed $= |\underline{v}| = 2\sqrt{26}a$ ms^{-1}
10. Both A and B follow a path with equation $y = x(x - 2)$.
    The acceleration of B is four times the acceleration of A.
11. When $t = \frac{5}{2}$ s and $\underline{r} = 25\underline{i} + \frac{75}{4}\underline{j}$
12. At $t = 0$ s where $\underline{r} = \underline{i}$

    and at $t = \frac{2}{3}$s where $\underline{r} = \frac{13}{9}\underline{i} - \frac{28}{27}\underline{j}$

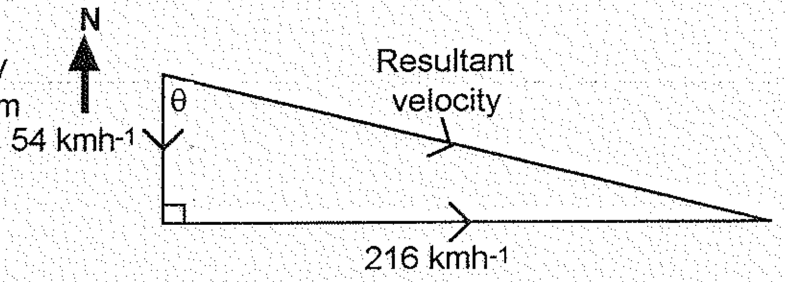
EFFECTS OF WIND AND CURRENTS

Example 1

A wind is blowing from the north at 54 kmh-1. A plane can fly at 216 kmh-1 in still air.
a) If the pilot steers due east, on what bearing will the plane travel?
b) What course should the pilot set in order to fly due east?
c) Calculate the actual flight speed of the plane.

Solution

The plane has two velocities, the velocity in still air and the velocity of the wind from the north.

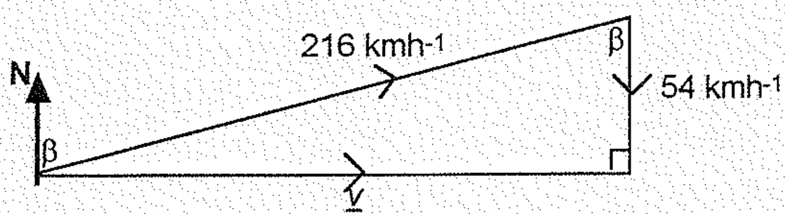


a) From sketch $tan\theta = \dfrac{216}{54} = 4$

   so $\theta \approx 76\cdot 0^0$

   Bearing travelled is 104^0

b) To fly due east the pilot will have to steer into the wind as shown.



   $cos\beta = \dfrac{54}{216} = 0\cdot 25$

   so $\beta \approx 75\cdot 5^0$

   The pilot will have to steer a bearing $075\cdot 5^0$.

c) The actual velocity of the plane is $\underline{v}$.

   $|\underline{v}|^2 = 216^2 - 54^2 = 43740$

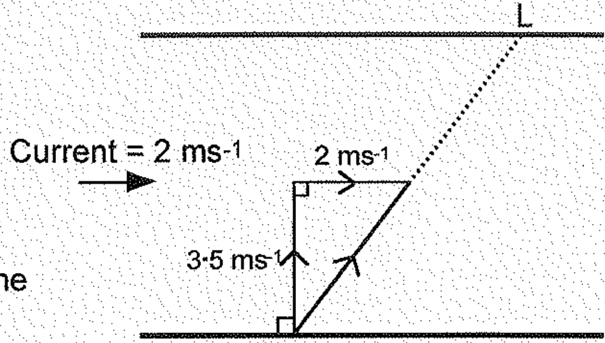
   Actual speed, $|\underline{v}| \approx 209$ kmh^{-1}.

Example 2

A boat can travel at 3·5 ms^{-1} in still water. A river is 100 m wide and the current flows at 2 ms^{-1}.
Calculate:
a) the shortest time taken to cross the river and the distance downstream the boat is carried
b) the course that must be set to cross the river to a point exactly opposite the starting point and the time taken for the crossing.

Solution

To cross in the shortest time the maximum component of velocity possible must be set directly across the river as shown.

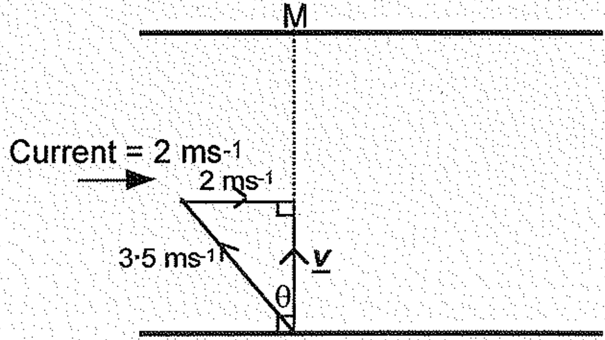


a) Time to cross $= \dfrac{\text{River width}}{3\cdot5} = \dfrac{100}{3\cdot5}$
$= 28\cdot57....$ s

Distance travelled $= 28\cdot57.... \times 2 \approx 57$ m downsteam to L

The time for the quickest crossing is $28\cdot6$ s and the distance travelled downstream ≈ 57 m.

To make a crossing to M, which is directly opposite the starting point, the boat must be headed upstream into the current as shown.



b) To cross to M the resultant velocity, $\underline{v}$, must be directed as shown.

$\sin\theta = \frac{2}{3\cdot5} = 0\cdot571.....$

so $\theta \approx 34\cdot8^0$

The boat has to be headed $34\cdot8^0$ upstream.

The actual speed of the boat is $|\underline{v}|$

and $|\underline{v}|^2 = 3\cdot5^2 - 2^2 = 8\cdot25$

so $|\underline{v}| = 2\cdot872...\ ms^{-1}$

Time to cross $= \dfrac{\text{River width}}{2\cdot872...} = \dfrac{100}{2\cdot872...}$
$\approx 34\cdot8$ s

The boat must be headed upstream by $34\cdot8^0$ to cross directly and the time taken will be $34\cdot8$ s.

Exercise M1.2-3

1. A man wishes to row across a river to reach a point on the far bank, exactly opposite his starting point. The river is 100 metres wide and flows at 3 ms^{-1}. In still water the man can row at 5 ms^{-1}. Find at what angle to the bank the man must steer the boat in order to complete the crossing, and the time taken.

2. A girl wishes to swim across a river, 100 metres wide, as quickly as possible. The river flows at 3 kmh^{-1} and the girl can swim at 4 kmh^{-1} in still water. Find the time that it takes the girl to cross the river and how far downstream she travels.

3. A pilot has to fly her light aircraft from airport A to airport B, 100 kilometres due east of A. In still air the aircraft flies at 125 kmh^{-1}. If there is a wind of 35 kmh^{-1} blowing from the north, find the course that the pilot must set in order to reach B and the time the journey takes.

4. Two airfields A and B are 500 kilometres apart with B on a bearing 060° from A. An aircraft which can travel at 200 kmh^{-1} in still air, is flown from A to B. If there is a wind of 40 kmh^{-1} blowing from the west, find the course that the pilot must set in order to reach B and find to the nearest minute, the time taken.

5. An aircraft capable of flying at 250 kmh^{-1} in still air, is flown from airport A to airport B, situated 300 kilometres from A on a bearing 320°. If there is a wind of 50 kmh^{-1} blowing from 030°, find the course the pilot must set and find, to the nearest minute, the time taken for the journey.

6. A man wishes to row a boat across a river to reach a point on the far bank that is 35 metres downstream from his starting point. The man can row at 2·5 ms^{-1} in still water. If the river is 50 metres wide and flows at 3 ms^{-1}, find **two** possible courses the man could set and find the respective crossing times.

7. When swimming in a river a man finds that he has a maximum speed *v* when swimming downstream and *u* when swimming upstream.
   a) Find an expression for his maximum speed when swimming in still water.
   b) If the river is of width *w*, show that the shortest time in which the man can swim across is $\frac{2w}{v+u}$ and that such a crossing would take him a distance of $\frac{w(v-u)}{v+u}$ downstream from his starting point.
   c) If the man wishes to swim quickly as possible from a point on one bank to a point exactly opposite on the other bank, show that he must swim in a direction that makes an angle $\cos^{-1}\left(\frac{v-u}{v+u}\right)$ with the bank and that the crossing will take a time $\frac{w}{\sqrt{uv}}$.

Answers 53·1° 90s

1. 53·3°; 25 s
2. 25 s; 75 m
3. 073·7°; 50 min
4. 054·3°; 2 h 8 min
5. 330·8°; 1 h 19 min
6. 45·6° to bank upstream or 24·4° to bank upstream; 28 s, 48 s
7. a) $\frac{v+u}{2}$

RELATIVE MOTION

Suppose a car is moving at 70 kmh^{-1} is being overtaken by a car moving at 80 kmh^{-1} on a motorway. To a passenger in the faster car who looks back, the slower car may seem to be moving backwards. To a passenger in the slower car, the faster car is pulling away from him slowly. The **speed** of the faster car **relative** to the slower car is

$$80 - 70 = 10\,\text{kmh}^{-1}$$

The speed of the slower car relative to the faster car is

$$70 - 80 = -10\,\text{kmh}^{-1}$$

If the cars had been going in opposite directions then their relative speed is

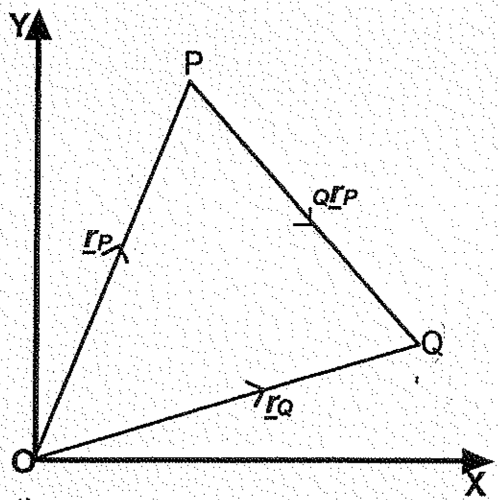
$$80 - (-70) = 150\,\text{kmh}^{-1}$$

which is also known as their **speed of approach.**

This example illustrates the idea of relative motion. There is a sense in which all motion is relative. When we stand still and watch objects move, we imagine we are at rest, when in fact the Earth, relative to which we are at rest, is both spinning on its axis and rotating around the Sun. These ideas were developed by Albert Einstein ("everything is relative").

In dealing with relative motion a way must be found of describing the position, velocity and acceleration of one body relative to another body. It is universally accepted now that the use of vectors is required, with the following notation being the one we shall use.

Relative position, velocity and acceleration



The diagram on the right shows the positions of two bodies P and Q and a fixed origin O.
With respect to O, P and Q have instantaneous position vectors $\underline{r}_P$ and $\underline{r}_Q$.
The instantaneous position vector of Q relative to P is $\vec{PQ}$.

Now $\vec{PQ} = \vec{PO} + \vec{OQ}$ so $\vec{PQ} = \underline{r}_Q - \underline{r}_P$

The notation we shall adopt is ${}_Q\underline{r}_P$ represents the position vector of Q relative to P

so ${}_Q\underline{r}_P = \underline{r}_Q - \underline{r}_P$

The corresponding expressions for velocity and acceleration are:

The velocity of Q relative to P is ${}_Q\underline{v}_P$ and ${}_Q\underline{v}_P = \underline{v}_Q - \underline{v}_P$.

The acceleration of Q relative to P is ${}_Q\underline{a}_P$ and ${}_Q\underline{a}_P = \underline{a}_Q - \underline{a}_P$.

Worked Example

A ship is sailing due west at 16 knots and a boat is sailing at 12 knots on a bearing of 340°. Find the velocity of the boat relative to the ship.

Solution

If $\underline{v}_S$ is the velocity of the ship and $\underline{v}_B$ is the velocity of the boat then the velocity of the boat relative to the velocity of the ship, ${}_B\underline{v}_S$, is $\underline{v}_B - \underline{v}_S$. Ways of evaluating $\underline{v}_B - \underline{v}_S$ include using *trigonometry* and *resolving the velocities into components*. Both methods will be considered on the next page.

Method 1 - Using Trigonometry

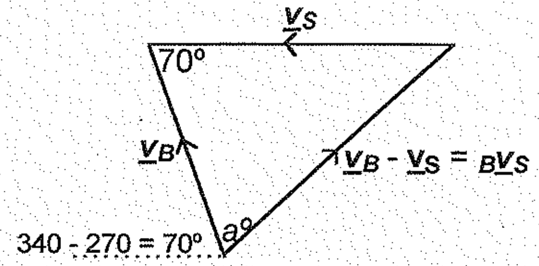
Using the Cosine Rule

$$|{}_B\underline{v}_S|^2 = |\underline{v}_S|^2 + |\underline{v}_B|^2 - 2|\underline{v}_S||\underline{v}_B|\cos 70^0$$

so $|{}_B\underline{v}_S|^2 = 16^2 + 12^2 - 2\times 16\times 12\times \cos 70^0$

$$|{}_B\underline{v}_S|^2 = 268\cdot 6\ldots$$

$$|{}_B\underline{v}_S| \approx 16\cdot 4 \text{ knots}$$



The velocity vector diagram shows the relative vector ${}_B\underline{v}_S$.

Using the Sine Rule

$$\frac{\sin \mathbf{a}^0}{16} = \frac{\sin 70^0}{16\cdot 3\ldots}$$

$$\sin \mathbf{a}^0 = 0\cdot 917\ldots$$

$$\mathbf{a}^0 \approx 66\cdot 5^0$$

The bearing angle of the relative velocity $= 340 + 66\cdot 5 - 360$

$$= 46\cdot 5^0$$

So the velocity of the boat relative to the ship is 16·4 knots on a bearing 046·5°.

Method 2 - Resolving Vectors

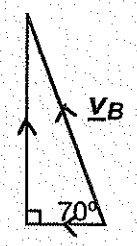
For the ship $\underline{v}_S = -16\underline{i}$ and for the boat $\underline{v}_B = -12\cos 70^0\,\underline{i} + 12\sin 70^0\,\underline{j}$

So ${}_B\underline{v}_S = \underline{v}_B - \underline{v}_S = (16 - 12\cos 70^0)\underline{i} + 12\sin 70^0\,\underline{j}$

and $|{}_B\underline{v}_S|^2 = (16 - 12\cos 70^0)^2 + (12\sin 70^0)^2$

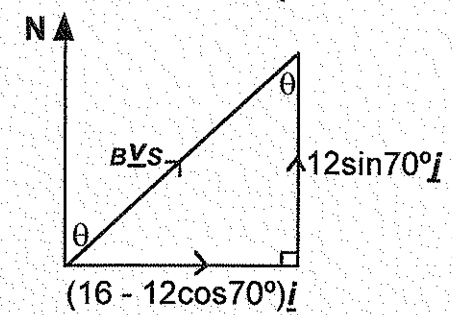
$$|{}_B\underline{v}_S|^2 = 286\cdot 6\ldots$$

$$|{}_B\underline{v}_S| \approx 16\cdot 4 \text{ knots}$$



From diagram opposite $\tan\theta = \dfrac{16 - 12\cos 70^0}{12\sin 70^0} = 1\cdot 054\ldots$

so $\theta \approx 46\cdot 5^0$



So again the velocity of the boat relative to the ship is 16·4 knots on a bearing 046·5°.

Exercise M1.2-4

1. A yacht and a trawler leave harbour. The yacht travels due west at 10 kmh^{-1} and the trawler due east at 20 kmh^{-1}. What is the velocity of the trawler relative to the yacht?
2. Particle A has a velocity of $(12\underline{i} + 5\underline{j})$ ms^{-1} and particle B has a velocity of $(4\underline{i} + 3\underline{j})$ ms^{-1}. Find the velocity of A relative to B.
3. A civil aircraft is moving with velocity $(300\underline{i} - 100\underline{j})$ kmh^{-1} and a fighter aircraft is moving with velocity $(400\underline{i} + 500\underline{j})$ kmh^{-1}. Find the velocity of the fighter relative to the civil aircraft.
4. The velocity of a particle P at time t is $6t\,\underline{i} - 2\underline{j}$. The velocity of Q relative to P at time t is $-4t\,\underline{i} + 5\underline{j}$. Find the velocity of Q at time t.
5. The velocity of a particle P is $2\underline{i} - 3t^2\underline{j}$. The velocity of another particle Q relative to P is $2t\,\underline{i} + 2\underline{j}$. If Q is initially at the point $\underline{i} - \underline{j}$, find the position vector of Q at time t.
6. Two particles P and Q start simultaneously from O with initial velocities $20\underline{i} + 30\underline{j}$ and $25\underline{i} + 15\underline{j}$ respectively and both move with acceleration $-10\underline{j}$.
   Find general expressions for ${}_Q\underline{a}_P$, ${}_Q\underline{v}_P$ and ${}_Q\underline{r}_P$.
7. A ship P is steaming N60°W at 15 kmh^{-1}. A ship Q is steaming due S at 25 kmh^{-1}.
   Produce two different solutions to find the magnitude and direction of ${}_Q\underline{v}_P$. (**As in notes**)
8. A ship steams due south at 20 kmh^{-1} while a power boat moves S60°E at 70 kmh^{-1}. Find the apparent velocity of the power boat to an observer on the ship.
9. A man is cycling along a level road at 6 ms^{-1}. Rain is falling. It falls vertically at 8 ms^{-1}. Find the apparent speed and direction of the rain to the cyclist.
10. To the same cyclist, as in Q.9, still riding along a level road at 6 ms^{-1} the inevitable rain appears to be falling at 12 ms^{-1}. If the rain is actually falling vertically, find the actual speed of the rain.
11. Three particles P, Q and R are moving with constant velocities. The velocities of P and Q are $\underline{i} + 3\underline{j}$ and $-4\underline{i} + 3\underline{j}$ respectively. The velocity of R relative to P is in the **direction** $3\underline{i} + 2\underline{j}$ and the velocity of R relative to Q is in the **direction** $7\underline{i} + 3\underline{j}$.
    Find the actual velocity of R.
12. A boat P is moving due east at 9 knots and a boat Q is moving N30°E at 6 knots. A boat R appears to an observer on P to be sailing due south, and to an observer on Q to be sailing S30°E.
    Find the actual velocity of R.
13. A man walking due east finds that the wind appears to blow from due north. When he doubles his velocity he finds that the wind appears to blow from NE.
    Find the direction from which the wind is actually blowing.

Answers

1. 30 kmh^{-1} east
2. $(8\underline{i} + 2\underline{j})$ ms^{-1}
3. $(100\underline{i} + 600\underline{j})$ kmh^{-1}
4. $\underline{v}_Q = 2t\,\underline{i} + 3\underline{j}$
5. $\underline{r}_Q = (t^2 + 2t + 1)\underline{i} + (-t^3 + 2t - 1)\underline{j}$
6. ${}_Q\underline{a}_P = \underline{0}$; ${}_Q\underline{v}_P = 5\underline{i} - 15\underline{j}$; ${}_Q\underline{r}_P = 5t\,\underline{i} - 15t\,\underline{j}$
7. 35 kmh^{-1}; 158·2°
8. 62·4 ms^{-1}; 103·9°
9. 10 ms^{-1}; 36·9° from vertical
10. 10·4 ms^{-1}
11. $10\underline{i} + 9\underline{j}$
12. 10·4 knots; 120°
13. From NW

COLLISION COURSES

In this class of problem we have two planes, ships or particles moving with **constant** velocity and given their initial positions we have to show that they will, if they continue to move with these velocities, collide. There are two contrasting ways of tackling problems of this type and both are illustrated in the following worked example.

Example 1

At noon the position vectors $\underline{r}$ and the velocity vectors $\underline{v}$ of two ships, A and B, are as follows:

$\underline{r}_A = (5\underline{i} - 3\underline{j})$ km $\quad\quad$ $\underline{v}_A = (2\underline{i} + 5\underline{j})$ kmh^{-1}

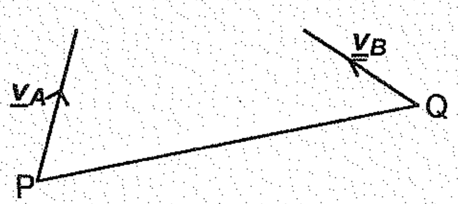
$\underline{r}_B = (7\underline{i} + 5\underline{j})$ km $\quad\quad$ $\underline{v}_B = (-3\underline{i} - 15\underline{j})$ kmh^{-1}

Show that if the velocities remain constant, a collision will occur, and find the time of the collision and the position vector of the point where it occurs.

Method 1

The idea behind this method is to simplify the problem so that only one of the ships is considered to be moving. It is then a straightforward process to see if the stationary ship is on the straight line route of the moving ship.

Consider two ships A and B initially at points P and Q.
A is moving with constant velocity $\underline{v}_A$ and B with constant velocity $\underline{v}_B$.



By imposing a velocity of $-\underline{v}_B$ on both A and B, then B can be considered at rest and the **resulting** velocity of A is $\underline{v}_A + (-\underline{v}_B) = \underline{v}_A - \underline{v}_B = {}_A\underline{v}_B$.

Thus if ${}_A\underline{v}_B$ is in the direction of PQ, the line joining the original position of A to that of B, then during the course of the motion A and B will collide. In other words, if A and B are to collide, A **must appear** to come straight towards B.

$\overrightarrow{PQ} = \underline{r}_B - \underline{r}_A = 7\underline{i} + 5\underline{j} - (5\underline{i} - 3\underline{j}) = 2\underline{i} + 8\underline{j}$

and ${}_A\underline{v}_B = \underline{v}_A - \underline{v}_B = 2\underline{i} + 5\underline{j} - (-3\underline{i} - 15\underline{j}) = 5\underline{i} + 20\underline{j} = \frac{5}{2}(2\underline{i} + 8\underline{j})$

So $\overrightarrow{PQ}$ is parallel to ${}_A\underline{v}_B$ and a collision will occur.

The displacement from P to Q is $2\underline{i} + 8\underline{j}$ so using $\underline{d} = t\underline{v}$

$2\underline{i} + 8\underline{j} = t(5\underline{i} + 20\underline{j})$ giving $t = \frac{2}{5}$ h $= 24$ min

The position of A (and B) when $t = \frac{2}{5}$ is $\underline{r} = 5\underline{i} - 3\underline{j} + t\underline{v}_A$

$\underline{r} = 5\underline{i} - 3\underline{j} + \frac{2}{5}(\overset{2}{\cancel{7}}\underline{i} + 5\underline{j}) = \frac{29}{5}\underline{i} - \underline{j}$.

The collision will occur at 12.24 pm at the point with position vector $\frac{29}{5}\underline{i} - \underline{j}$.

Method 2

The second method has a much simpler basic idea. A collision will occur if the two ships are at the same point at the same time. The solution requires general expressions for $\underline{r}_A$ and $\underline{r}_B$, as functions of time, to be found.

Starting with the constant velocities:

$\underline{v}_A = 2\underline{i} + 5\underline{j} \quad\quad \underline{v}_B = -3\underline{i} - 15\underline{j}$

Integrating gives $\underline{r}_A = 2t\underline{i} + 5t\underline{j} + \underline{c}_1 \quad\quad \underline{r}_B = -3t\underline{i} - 15t\underline{j} + \underline{c}_2$

When $t = 0$ $\underline{r}_A = 5\underline{i} - 3\underline{j}$ so $\underline{c}_1 = 5\underline{i} - 3\underline{j}$ and $\underline{r}_B = 7\underline{i} + 5\underline{j}$ so $\underline{c}_2 = 7\underline{i} + 5\underline{j}$

Giving $\underline{r}_A = (2t + 5)\underline{i} + (5t - 3)\underline{j}$ and $\underline{r}_B = (7 - 3t)\underline{i} + (5 - 15t)\underline{j}$

For a collision to occur $\underline{r}_A = \underline{r}_B$ at some instant in time where $t \geq 0$.

So for a collision $(2t+5)\underline{i} + (5t-3)\underline{j} = (7-3t)\underline{i} + (5-15t)\underline{j}$

$$\begin{aligned} \text{Giving } 2t+5 &= 7-3t \quad \text{and} \quad 5t-3 = 5-15t \\ 5t &= 2 \quad \text{and} \quad 20t = 8 \\ t &= \tfrac{2}{5} \quad \text{and} \quad t = \tfrac{2}{5} \end{aligned}$$

So the position vectors of the ships are the same after $\frac{2}{5}$h = 24 min at 12.24 pm.

The position vector of the collision is $(2 \times \frac{2}{5} + 5)\underline{i} + (5 \times \frac{2}{5} - 3)\underline{j}$

$= \frac{29}{5}\underline{i} - \underline{j}$ as before.

Example 2

At $t = 0$ s a particle A starts from the point $\underline{i} - 4\underline{j}$ and moves with constant velocity $(\frac{1}{3}\underline{i} + \frac{1}{2}\underline{j})$ ms^{-1}. Simultaneously, a particle B starts from the point $-3\underline{i} + 5\underline{j}$ and moves with constant velocity $(\underline{i} + a\underline{j})$ ms^{-1}. If A and B collide, find the time of the collision, the value of a and the position vector of the point of collision.

Starting with the constant velocities:

$$\underline{v}_A = \tfrac{1}{3}\underline{i} + \tfrac{1}{2}\underline{j} \qquad \underline{v}_B = \underline{i} + a\underline{j}$$

Integrating gives $\underline{r}_A = \frac{1}{3}t\underline{i} + \frac{1}{2}t\underline{j} + \underline{c}_1$ $\qquad$ $\underline{r}_B = t\underline{i} + at\underline{j} + \underline{c}_2$

When $t = 0$ $\underline{r}_A = \underline{i} - 4\underline{j}$ so $\underline{c}_1 = \underline{i} - 4\underline{j}$ and $\underline{r}_B = -3\underline{i} + 5\underline{j}$ so $\underline{c}_2 = -3\underline{i} + 5\underline{j}$

Giving $\underline{r}_A = (\frac{1}{3}t + 1)\underline{i} + (\frac{1}{2}t - 4)\underline{j}$ and $\underline{r}_B = (t-3)\underline{i} + (at+5)\underline{j}$

For a collision to occur $\underline{r}_A = \underline{r}_B$ at some instant in time where $t \geq 0$.

So for a collision $(\frac{1}{3}t + 1)\underline{i} + (\frac{1}{2}t - 4)\underline{j} = (t-3)\underline{i} + (at+5)\underline{j}$

$$\begin{aligned} \text{Giving } \tfrac{1}{3}t + 1 &= t - 3 \quad \text{and} \quad \tfrac{1}{2}t - 4 = at + 5 \\ -\tfrac{2}{3}t &= -4 \\ t &= 6 \text{ s} \end{aligned}$$

Substituting for t in the y component equation gives $\frac{1}{2} \times 6 - 4 = 6a + 5$

so $a = -1$

So the position vectors of the particles are the same after 6 s.

The position vector of the collision is $(\frac{1}{3} \times 6 + 1)\underline{i} + (\frac{1}{2} \times 6 - 4)\underline{j}$

$= 3\underline{i} - \underline{j}$

Exercise M1.2-5

1. Repeat **Example 2** on page 28 using the approach outlined in **method 1** on page 27.

2. Initially two boats A and B are 100 metres apart with A due east of B. A has a constant velocity of $(2\underline{i} + 3\underline{j})$ ms^{-1} and B a constant speed of 5 ms^{-1}. Find, in vector form, the velocity of B if it is to intercept A and find the time taken to do so.

3. Initially two boats A and B are 48 metres apart with B due north of A. A has a constant velocity of $(5\underline{i} + 4\underline{j})$ ms^{-1} and B a constant speed of 13 ms^{-1}. Find, in vector form, the velocity of B if it is to intercept A and find the time taken to do so.

4. At noon the position vectors $\underline{r}$ and the velocity vectors $\underline{v}$ of two ships, A and B, are as follows:

   $\underline{r}_A = (\underline{i} + 7\underline{j})$ km $\qquad \underline{v}_A = (6\underline{i} + 2\underline{j})$ kmh^{-1}

   $\underline{r}_B = (6\underline{i} + 4\underline{j})$ km $\qquad \underline{v}_B = (-4\underline{i} + 8\underline{j})$ kmh^{-1}

   Show that if the velocities remain constant, a collision will occur, and find the time of the collision and the position vector of the point where it occurs.

5. At noon the position vectors $\underline{r}$ and the velocity vectors $\underline{v}$ of two ships, A and B, are as follows:

   $\underline{r}_A = (5\underline{i} + 2\underline{j})$ km $\qquad \underline{v}_A = (15\underline{i} + 10\underline{j})$ kmh^{-1}

   $\underline{r}_B = (7\underline{i} + 7\underline{j})$ km $\qquad \underline{v}_B = (9\underline{i} - 5\underline{j})$ kmh^{-1}

   Show that if the velocities remain constant, a collision will occur, and find the time of the collision and the position vector of the point where it occurs.

6. At 11.30 am a naval frigate is at a place with position vector $(-6\underline{i} + 12\underline{j})$ km and is moving with velocity $(16\underline{i} - 4\underline{j})$ kmh^{-1}. At **noon** a naval destroyer is at a place with position vector $(12\underline{i} - 15\underline{j})$ km and is moving with velocity $(8\underline{i} + 16\underline{j})$ kmh^{-1}. Show that if these velocities are maintained the two ships will collide and find when and where the collision occurs.

7. At noon a passenger jet has a position vector $(-100\underline{i} + 220\underline{j} + 10\underline{k})$ km and a velocity vector $(300\underline{i} + 400\underline{j})$ kmh^{-1}. At 12.15 pm another passenger jet has a position vector $(-60\underline{i} + 355\underline{j} + 10\underline{k})$ km and a velocity vector $(400\underline{i} + 300\underline{j})$ kmh^{-1}. Show that if these velocities are maintained the jets will crash into each other and find the time and position vector of the crash.

8. At a certain instant two particles P and Q are at points A and B which are 150 centimetres apart with B due east of A. Particle P is travelling at $10\sqrt{3}$ cms^{-1} due south and Q is travelling at 20 cms^{-1} in a direction S30°W. Show that if the velocities remain unchanged, a collision will take place and find the time which elapses before it does so.

Answers

1. 6 s; $a = -1$; $3\underline{i} - \underline{j}$
2. $(4\underline{i} + 3\underline{j})$ ms^{-1}; 50 s
3. $(5\underline{i} - 12\underline{j})$ ms^{-1}; 3 s
4. 12.30 pm; $(4\underline{i} + 8\underline{j})$ km
5. 12.20 pm; $(10\underline{i} + \frac{16}{3}\underline{j})$ km
6. 1.15 pm; $(22\underline{i} + 5\underline{j})$ km
7. 12.36 pm; $(80\underline{i} + 460\underline{j} + 10\underline{k})$ km
8. 15 s

NEAREST APPROACH

In this class of problem we have two planes, ships or particles moving with **constant** velocity and given their initial positions we have to determine when they will be closest to one another. As in the last section there are two contrasting ways of tackling problems of this type and both are illustrated in the following worked example.

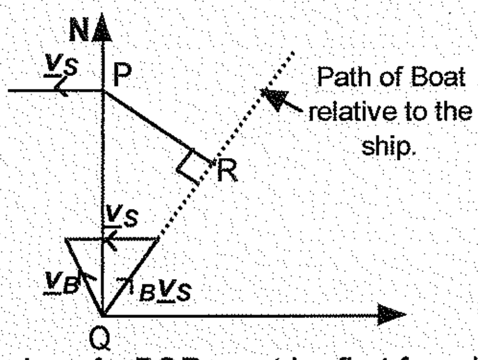
Worked Example

At noon a ship is 10 nautical miles due north of a boat. The ship is sailing due west at 16 knots and the boat is sailing at 12 knots on a bearing of 340°. Find the time when they are closest together and their distance apart at the time.

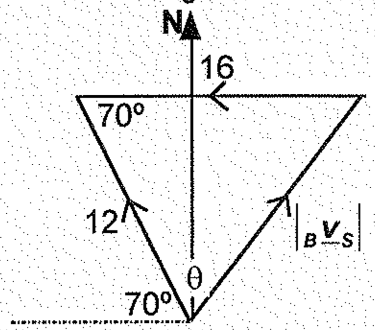
Method 1

The idea behind this method is to simplify the problem so that only one of the objects is considered to be moving. To an observer on the ship the ship appears to be stationary and the boat is travelling with velocity ${}_B\underline{v}_S = \underline{v}_B - \underline{v}_S$.

From the first diagram on the left it can be seen that the boat is at its closest position to the ship when the boat is at R on the path of the boat relative to the ship. The distance between the boat and ship at this point is PR.



Velocity vector diagram



To be able to calculate PR the size of ∠PQR must be first found.

From the velocity vector diagram using the cosine rule:

$$\left|{}_B\underline{v}_S\right|^2 = 12^2 + 16^2 - 2\times 12\times 16\cos 70^0$$

$$\left|{}_B\underline{v}_S\right| = 16\cdot 39\ldots$$

Now using the sine rule:

$$\frac{\sin\theta^0}{16} = \frac{\sin 70^0}{\left|{}_B\underline{v}_S\right|}$$

$$\sin\theta^0 = \frac{\sin 70^0}{\left|{}_B\underline{v}_S\right|}\times 16$$

$$\sin\theta^0 = 0\cdot 917\ldots$$

$$\theta^0 \approx 66\cdot 5^0$$

So $\angle PQR = 66\cdot 5 - 20 = 46\cdot 5^0$

Using SOH CAH TOA

$$\sin 46\cdot 5^0 = \frac{PR}{PQ}$$

so $PR = PQ\times\sin 46\cdot 5^0 = 10\times\sin 46\cdot 5^0$

$PR \approx 7\cdot 25$ nautical miles

Using $T = \dfrac{D}{S}$ then time taken $= \dfrac{QR}{\left|{}_B\underline{v}_S\right|}$

$$= \frac{10\times\cos 46\cdot 5^0}{16\cdot 39\ldots}$$

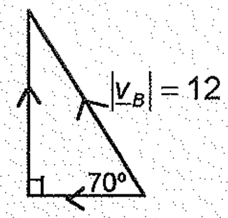
$$= 0\cdot 420\ldots\text{ h}$$

$$\approx 25\text{ min}$$

Boat is closest to the ship at 12.25 pm.

Method 2

With this method an expression for ${}_B\underline{r}_S$ is found and then, with the aid of calculus, the minimum value of $\left|{}_B\underline{r}_S\right|$ is calculated.



For the ship $\underline{v}_S = -16\underline{i}$ and for the boat $\underline{v}_B = -12\cos 70^0\,\underline{i} + 12\sin 70^0\,\underline{j}$

Starting with the constant velocities

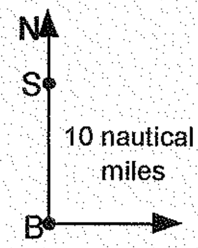
$\underline{v}_S = -16\underline{i}$ $\qquad\qquad$ $\underline{v}_B = -12\cos 70^0\,\underline{i} + 12\sin 70^0\,\underline{j}$

Integrating gives

$\underline{r}_S = -16t\,\underline{i} + \underline{c}_1$ $\qquad\qquad$ $\underline{r}_B = -12t\cos 70^0\,\underline{i} + 12t\sin 70^0\,\underline{j} + \underline{c}_2$

When $t = 0$ $\underline{r}_S = 10\underline{j}$ so $\underline{c}_1 = 10\underline{j}$ and $\underline{r}_B = \underline{0}$ so $\underline{c}_2 = \underline{0}$

Giving $\underline{r}_S = -16t\,\underline{i} + 10\underline{j}$ and $\underline{r}_B = -12t\cos 70^0\,\underline{i} + 12t\sin 70^0\,\underline{j}$



Now using ${}_B\underline{r}_S = \underline{r}_B - \underline{r}_S$

${}_B\underline{r}_S = (16t - 12t\cos 70^0)\underline{i} + (12t\sin 70^0 - 10)\underline{j}$

Giving $\left|{}_B\underline{r}_S\right|^2 = (16 - 12\cos 70^0)^2 t^2 + (12t\sin 70^0 - 10)^2$

$\left|{}_B\underline{r}_S\right|^2 \approx 268\cdot 7t^2 - 225\cdot 5t + 100$

This is a quadratic expression in t which has a minimum value.

$\left|{}_B\underline{r}_S\right|^2$ has its minimum at the same time as $\left|{}_B\underline{r}_S\right|$ has its minimum

and it is easier to find $\dfrac{d\left|{}_B\underline{r}_S\right|^2}{dt}$ than $\dfrac{d\left|{}_B\underline{r}_S\right|}{dt}$. [TRY IT AND BE CONVINCED]

So $\dfrac{d\left|{}_B\underline{r}_S\right|^2}{dt} = 537\cdot 4t - 225\cdot 5$

The minimum value of $\left|{}_B\underline{r}_S\right|^2$ occurs when $537\cdot 4t - 225\cdot 5 = 0$

Giving $t = \dfrac{225\cdot 5}{537\cdot 4} \approx 0\cdot 420\text{ h} = 25\text{ min}$

Closest together at 12.25 pm

At this time $\left|{}_B\underline{r}_S\right|^2 = 268\cdot 7 \times (0\cdot 420)^2 - 225\cdot 5 \times 0\cdot 420 + 100$

$\left|{}_B\underline{r}_S\right|^2 = 52\cdot 6\ldots$

$\left|{}_B\underline{r}_S\right| \approx 7\cdot 25$ nautical miles as before.

Exercise M1.2-6

1. At noon the position vectors $\underline{r}$ and the velocity vectors $\underline{v}$ of two ships, A and B, are as follows:

$$\underline{r}_A = (-9\underline{i} + 6\underline{j}) \text{ km} \qquad \underline{v}_A = (3\underline{i} + 12\underline{j}) \text{ kmh}^{-1}$$
$$\underline{r}_B = (16\underline{i} + 6\underline{j}) \text{ km} \qquad \underline{v}_B = (-9\underline{i} + 3\underline{j}) \text{ kmh}^{-1}$$

   a) Find how far apart the ships are at noon.
   b) Assuming the velocities do not change, find the least distance between the ships in the subsequent motion.
   c) Find when this distance of closest approach occurs and the position vectors of A and B at that time.

2. At 9 am two ships A and B are 11 kilometres apart with B due west of A. A and B travel with constant velocities of $(-4\underline{i} + 3\underline{j})$ kmh^{-1} and $(2\underline{i} + 4\underline{j})$ kmh^{-1} respectively. Find the least distance between the two ships in the subsequent motion and the time, to the nearest minute, at which this situation occurs.

3. Two ships A and B have the following position vectors $\underline{r}$ and velocity vectors $\underline{v}$ at the times indicated:

$\underline{r}_A = (3\underline{i} + \underline{j})$ km	$\underline{v}_A = (2\underline{i} + 3\underline{j})$ kmh^{-1}	at noon
$\underline{r}_B = (2\underline{i} - \underline{j})$ km	$\underline{v}_B = (3\underline{i} + 7\underline{j})$ kmh^{-1}	at 1.00 pm.

   Assuming the ships maintain these velocities, find:
   a) the position vector of ship A at 1.00 pm
   b) the distance between A and B at 1.00 pm
   c) the least distance between A and B during the motion
   d) the time, to the nearest minute, when the least distance occurs.

4. Two aircraft A and B are flying at the same altitude with velocities 180 ms^{-1} due east and 240 ms^{-1} due north respectively. Initially B is 5 kilometres due south of A. Given that the aircraft do not change their velocities, find the shortest distance between the aircraft in the subsequent motion and the time taken for that situation to occur.

5. A road running north to south crosses a road running east to west at a junction O. John cycles towards O from the west at 3 ms^{-1} as Robert cycles towards O from the south at 4 ms^{-1}. Initially John is 600 metres from O when Robert is 250 metres from O. If John and Robert do not alter their velocities, find the least distance they are apart during the motion and the time taken to reach that situation.

6. Speedboat B is travelling at a constant velocity of 10 ms^{-1} due east and speedboat A is travelling at a constant speed of 8 ms^{-1}. Initially A and B are 600 metres apart with A due south of B. Find the course that A should set in order to get as close as possible to B. Find this closest distance and the time taken for the situation to occur.

7. At noon two ships A and B are 5·2 kilometres apart with A due west of B, and B is travelling due north at a steady 13 kmh^{-1}. If A travels with a constant speed of 12 kmh^{-1} show that, for A to get as close as possible to B, A should set a course of Nθ^0E where $sin\,\theta^0 = \frac{5}{12}$. 13.
   Find this closest distance and the time at which it occurs.

Answers

1. a) 25 km b) 15 km c) 1.20 pm; $(-5\underline{i} + 22\underline{j})$ km; $(4\underline{i} + 10\underline{j})$ km   2. 1·81 km; 10.47 pm
3. a) $(5\underline{i} + 4\underline{j})$ km b) 5·83 km c) 1·70 km d) 2.21 pm   4. 3 km; $13\frac{1}{3}$ s   5. 330 m; 112 s
6. 053·3°; 360 m; 80 s   7. 2 km; ≈ 12.58 pm

M1.3 MOTION OF PROJECTILES IN A VERTICAL PLANE

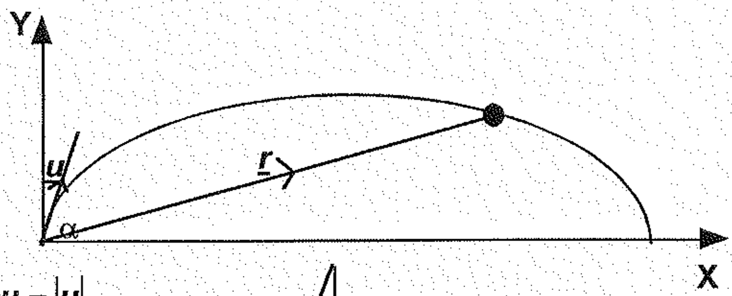
Outcome Content
Know the meaning of the terms projectiles, velocity and angle of projection, trajectory, time of flight, range and constant gravity.
Solve the vector equation $\ddot{\underline{r}} = -g\underline{j}$ to obtain $\underline{r}$ in terms of its horizontal and vertical components.
Obtain and solve the equations of motion $\ddot{x} = 0$ and $\ddot{y} = -g$, obtaining expressions for $\dot{x}$, $\dot{y}$, x and y in any particular case.
Find the time of flight, the greatest height reached and the range of a projectile.
Find the maximum range of a projectile on a horizontal plane and the angle of projection to achieve this.
Find, and use, the equation of the trajectory of a projectile.
Solve problems in two-dimensional motion involving projectiles under constant gravitational force and neglecting air resistance.

Projectiles are bodies that are thrown, dropped or launched into the air under the influence of gravity only. Projectiles have no means of controlling their motion like, for example, a rocket.

In modelling the motion of a projectile certain assumptions are made, viz., that the motion is under **constant gravity** and the air resistance can be ignored. Furthermore, we consider the body to be a particle and so ignore any aerodynamic effects due to the shape of the body. Gravity acts vertically, if the unit vector $\underline{j}$ is taken to have direction vertically upwards then the motion of the projected particle is described by the vector equation $\ddot{\underline{r}} = -g\underline{j}$.

Solving the vector equation $\ddot{\underline{r}} = -g\underline{j}$.

A projectile is launched from the origin with initial velocity $\underline{u}$ and **angle of projection** α^0.



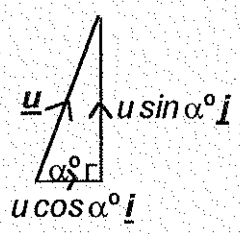
$\ddot{\underline{r}} = -g\underline{j}$

Integrating gives

$\dot{\underline{r}} = -gt\underline{j} + \underline{c}_1$

When $t = 0$, $\dot{\underline{r}} = u\cos\alpha^0\underline{i} + u\sin\alpha^0\underline{j}$ where $u = |\underline{u}|$

So $\dot{\underline{r}} = u\cos\alpha^0\underline{i} + (u\sin\alpha^0 - gt)\underline{j}$ [1]



Integrating again gives

$\underline{r} = ut\cos\alpha^0\underline{i} + (ut\sin\alpha^0 - \frac{1}{2}gt^2)\underline{j} + \underline{c}_2$

When $t = 0$, $\underline{r} = \underline{0}$ so $\underline{c}_2 = \underline{0}$ giving

$\underline{r} = ut\cos\alpha^0\underline{i} + (ut\sin\alpha^0 - \frac{1}{2}gt^2)\underline{j}$ [2]

Comparing [1] with $\dot{\underline{r}} = \dot{x}\underline{i} + \dot{y}\underline{j}$ we have $\dot{x} = u\cos\alpha^0$ and $\dot{y} = u\sin\alpha^0 - gt$

Comparing [2] with $\underline{r} = x\underline{i} + y\underline{j}$ we have $x = ut\cos\alpha^0$ and $y = ut\sin\alpha^0 - \frac{1}{2}gt^2$

Time of flight

The projectile is in flight as long as $y > 0$. When $y = 0$ the projectile is either at the point of being launched or is returning to the level ground. This simple fact can be used to determine the time of flight.

Starting with $y = ut\sin\alpha^0 - \frac{1}{2}gt^2$

If $y = 0$ then $ut\sin\alpha^0 - \frac{1}{2}gt^2 = 0$

so $t(u\sin\alpha^0 - \frac{1}{2}gt) = 0$

giving $t = 0$ or $u\sin\alpha^0 - \frac{1}{2}gt = 0$

Hence $y = 0$ when $t = 0$ or $t = \frac{2u\sin\alpha^0}{g}$

$t = 0$ relates to the instant of projection while $t = \frac{2u\sin\alpha^0}{g}$ is the time of the flight.

Range of the projectile

The horizontal distance travelled by a projectile is called its range.

The time of the flight is $\frac{2u\sin\alpha^0}{g}$ and we found that $x = ut\cos\alpha^0$.

When $t = \frac{2u\sin\alpha^0}{g}$, $x = u \times \frac{2u\sin\alpha^0}{g} \times \cos\alpha^0 = \frac{2u^2\sin\alpha^0\cos\alpha^0}{g}$

but $\sin 2\alpha^0 = 2\sin\alpha^0\cos\alpha^0$ so $x = \frac{u^2\sin 2\alpha^0}{g}$

and the range of the projectile on the horizontal plane is $R = \frac{u^2\sin 2\alpha^0}{g}$.

Note that when $t = \frac{2u\sin\alpha^0}{g}$, and substituting into our earlier result for $\dot{y}$

gives $\dot{y} = u\sin\alpha^0 - g\frac{2u\sin\alpha^0}{g} = -u\sin\alpha^0$ showing that the projectile hits the ground with a speed equal to its initial speed.

The maximum range on the horizontal plane

From the formula for the range of the projectile it follows for a given value of u the maximum range will occur when $\sin 2\alpha^o$ is a maximum. The sine function has a maximum value of one when the angle is 90°. So for the maximum range $2\alpha = 90$, thus the maximum range on the horizontal plane is given by $R_{max} = \frac{u^2}{g}$, when $\alpha = 45$.

The greatest height

The greatest height will be reached when the vertical component of velocity is zero.

Using the earlier result for $\dot{y}$. If $\dot{y} = 0$ then $u\sin\alpha^0 - gt = 0$

which gives $t = \frac{u\sin\alpha^0}{g}$.

Using $y = ut\,sin\,\alpha^0 - \frac{1}{2}gt^2$ and substituting for t

gives $H = u \times \frac{u\,sin\,\alpha^0}{g} \times sin\,\alpha^0 - \frac{1}{2}g \times \left(\frac{u\,sin\,\alpha^0}{g}\right)^2$

$$H = \frac{u^2\,sin^2\,\alpha^0}{g} - \frac{1}{2}\frac{u^2\,sin^2\,\alpha^0}{g}$$

$$H = \frac{u^2\,sin^2\,\alpha^0}{2g}$$

Note, the time to reach the greatest height $= \frac{u\,sin\,\alpha^0}{g} = \frac{1}{2}\frac{2u\,sin\,\alpha^0}{g} = \frac{1}{2}$ time for whole flight.

The shape of the trajectory

The trajectory of the projectile is the path traced out during the flight. We found earlier when solving the vector equation $\ddot{\underline{r}} = -g\underline{j}$, that the solution for $\underline{r}$ had the components

$$x = ut\,cos\,\alpha^0 \quad ..... \quad [3] \quad \text{and} \quad y = ut\,sin\,\alpha^0 - \frac{1}{2}gt^2 \quad ..... \quad [4]$$

The equations [3] and [4] are known as the parametric equations of the trajectory, with parameter t.

To obtain the mathematical relationship between x and y we have to remove t by substitution.

Rearranging [3] gives $t = \frac{x}{u\,cos\,\alpha^0}$, then substituting for t in [4] gives

$$y = u \times \frac{x}{u\,cos\,\alpha^0} \times sin\,\alpha^0 - \frac{1}{2}g \times \left(\frac{x}{u\,cos\,\alpha^0}\right)^2 \qquad \text{But } \frac{sin\,\alpha^0}{cos\,\alpha^0} = tan\,\alpha^0$$

$$\text{So } y = x\,tan\,\alpha^0 - \frac{g}{2u^2\,cos^2\,\alpha^0}x^2 \quad ..... \quad [5]$$

This is the Cartesian equation of the trajectory of the projectile. It is of the form $y = ax^2 + bx$ which is the equation of a parabola.

The equation $y = x\,tan\,\alpha^0 - \frac{g}{2u^2\,cos^2\,\alpha^0}x^2$ provides another means of finding an expression for the range of the projectile on the horizontal plane. The projectile meets the horizontal plane at two points, the point of projection and the point of return. Both points have $y = 0$.

So at these points $x\,tan\,\alpha^0 - \frac{g}{2u^2\,cos^2\,\alpha^0}x^2 = 0$

$$\text{giving } x\left(tan\,\alpha^0 - \frac{g}{2u^2\,cos^2\,\alpha^0}x\right) = 0$$

$$\text{so } x = 0 \text{ or } tan\,\alpha^0 - \frac{g}{2u^2\,cos^2\,\alpha^0}x = 0$$

$$x = 0 \text{ or } x = \frac{2u^2\,cos^2\,\alpha^0\,tan\,\alpha^0}{g} = \frac{u^2\,sin\,2\alpha^0}{g} \text{ as before.}$$

The point of projection is where $x = 0$ and the point of return gives the range $R = \frac{u^2\,sin\,2\alpha^0}{g}$.

The shape of the trajectory continued

Before we can continue a new trigonometrical identity has to be established, but first it has to be pointed out that there are three additional trigonometrical functions. These are:

$$cosec\, x^0 = \frac{1}{sin\, x^0}; \quad sec\, x^0 = \frac{1}{cos\, x^0}; \quad cotan\, x^0 = \frac{1}{tan\, x^0}$$

The equation of the trajectory contains $\frac{1}{cos^2\alpha^0}$. It is found convenient to replace this with $sec^2\alpha^0$.

Starting with the identity first met in fourth year, $sin^2 x^0 + cos^2 x^0 = 1$, and dividing through by $cos^2 x^0$ gives $\frac{sin^2 x^0}{cos^2 x^0} + \frac{cos^2 x^0}{cos^2 x^0} = \frac{1}{cos^2 x^0}$. Using the other identity also met in fourth year, $tan\, x^0 = \frac{sin\, x^0}{cos\, x^0}$, and also substituting for $\frac{1}{cos\, x^0}$ gives the combined identity:

$$tan^2 x^0 + 1 = sec^2 x^0$$

The equation of the trajectory, $y = x\, tan\,\alpha^0 - \frac{g}{2u^2 cos^2\alpha^0}x^2$ can now be written as:

$$y = x\, tan\,\alpha^0 - \frac{g\, sec^2\alpha^0}{2u^2}x^2$$

and since $sec^2\alpha^0 = 1 + tan^2\alpha^0$ the last equation becomes:

$$y = x\, tan\,\alpha^0 - \frac{g}{2u^2}(1 + tan^2\alpha^0)x^2$$

In this form, if we are given values for x, y and u, we have a quadratic equation in $tan\,\alpha^0$. The two solutions for α, $0 < \alpha < 90$, give the two angles of projection of the projectile with initial speed u required for the projectile to pass through the point (x,y).

SOLVING PROBLEMS INVOLVING PROJECTILES

There are three possible approaches to tackling problems involving projectiles.

1. From first principles, starting from either $\ddot{\underline{r}} = -g\underline{j}$ or $\ddot{x} = 0$ and $\ddot{y} = -g$.
2. Working with the horizontal and vertical components of the motion separately, using speed-distance-time for the horizontal direction and Newton's equations of motion with constant acceleration in the the vertical direction.
3. For appropriate parts, quoting and using formulae established in the last four pages.

All these approaches are acceptable, with the first one being the favoured approach.

WORKED EXAMPLES

Example 1

A golf ball is struck with initial velocity $25\underline{i} + 15\underline{j}$, measured in ms^{-1}, where $\underline{i}$ and $\underline{j}$ are horizontal and vertical unit vectors. Find:

a) its velocity $\underline{v}$ after t seconds
b) its position vector $\underline{r}$ at this instant
c) its time of flight before pitching on to a horizontal fairway
d) its horizontal range
e) its maximum height. [Assume that $g = 10\ ms^{-2}$]

Solution

Method 1

a) The acceleration, $\underline{a}$, at any instant is $-g\underline{j}$.

Hence its velocity, t seconds after being hit, is given by $\dfrac{d\underline{v}}{dt} = -g\underline{j}$

Integrating gives $\underline{v} = -gt\,\underline{j} + \underline{c}_1$

Now $\underline{v} = 25\underline{i} + 15\underline{j}$ when $t = 0$, so $\underline{c}_1 = 25\underline{i} + 15\underline{j}$

Giving $\underline{v} = 25\underline{i} + (15 - gt)\underline{j}$

b) Using $\underline{v} = 25\underline{i} + (15 - gt)\underline{j}$

Integrating gives $\underline{r} = 25t\underline{i} + (15t - \frac{1}{2}gt^2)\underline{j} + \underline{c}_2$

Now $\underline{r} = \underline{0}$ when $t = 0$, so $\underline{c}_2 = \underline{0}$

Giving $\underline{r} = 25t\underline{i} + (15t - \frac{1}{2}gt^2)\underline{j}$

c) The golf ball stikes the fairway when the y component of $\underline{r}$ is zero.

i.e. when $15t - \frac{1}{2}gt^2 = 0$

so $t(15 - \frac{1}{2} \times 10t) = 0$

giving $t = 0$ or $t = 3$

The golf ball strikes the fairway after 3 seconds.

d) The x component of $\underline{r}$ at $t = 3$ gives the range of the golf ball.

Range = $25 \times 3 = 75$ metres.

e) The maximum height occurs when the vertical component of $\underline{v}$ is zero.

At maximum height when $15 - gt = 0$

so $15 - 10 \times t = 0$

giving $t = 1{\cdot}5$

The golf ball is at its maximum height after $1{\cdot}5$ seconds.

The y component of $\underline{r}$ at $t = 1{\cdot}5$ gives the maximum height of the golf ball.

Maximum height = $15 \times 1{\cdot}5 - \frac{1}{2} \times 10 \times (1{\cdot}5)^2 = 11{\cdot}25$ metres

Method 2

a) The horizontal component of velocity is constant since there is no horizontal force acting in our model.

Horizontal component of velocity $= 25\ \text{ms}^{-1}$

Since the golf ball is moving with constant acceleration $-\boldsymbol{g}$ vertically, the vertical component of velocity can be obtained by using $\boldsymbol{v} = \boldsymbol{u} + \boldsymbol{at}$ where $\boldsymbol{u} = 15$ and $\boldsymbol{a} = -\boldsymbol{g} = -10$.

So vertical component of velocity $= 15 - 10\boldsymbol{t}$

giving $\underline{\boldsymbol{v}} = 25\underline{\boldsymbol{i}} + (15 - 10\boldsymbol{t})\underline{\boldsymbol{j}}$

b) The distance travelled horizotally in time $\boldsymbol{t}$ is obtained using $\boldsymbol{s} = \boldsymbol{ut} + \frac{1}{2}\boldsymbol{at}^2$ where $\boldsymbol{a} = 0$.

So the distance travelled horizontally in $\boldsymbol{t}$ seconds $= 25\boldsymbol{t}$ metres.

The distance travelled vertically in time $\boldsymbol{t}$ is obtained using $\boldsymbol{s} = \boldsymbol{ut} + \frac{1}{2}\boldsymbol{at}^2$

where $\boldsymbol{a} = -\boldsymbol{g} = -10$.

So the distance travelled vertically in $\boldsymbol{t}$ seconds $= 15\boldsymbol{t} - 5\boldsymbol{t}^2 = 5\boldsymbol{t}(3 - \boldsymbol{t})$.

These two distances form the components of $\underline{\boldsymbol{r}}$.

Giving $\underline{\boldsymbol{r}} = 25\boldsymbol{t}\underline{\boldsymbol{i}} + 5\boldsymbol{t}(3 - \boldsymbol{t})\underline{\boldsymbol{j}}$

c) The vertical distance $= 0$ when $5\boldsymbol{t}(3 - \boldsymbol{t}) = 0$

so $\boldsymbol{t} = 0$ or $\boldsymbol{t} = 3$

The golf ball strikes the fairway after 3 seconds.

d) The range $=$ The distance travelled horizontally in 3 seconds $= 25 \times 3 = 75$ metres.

e) The maximum height occurs when the vertical component of velocity is zero.

At maximum height when $15 - 10\boldsymbol{t} = 0$

Giving $\boldsymbol{t} = 1{\cdot}5$

Maximum height $= 15 \times 1{\cdot}5 - 5 \times (1{\cdot}5)^2 = 11{\cdot}25$ metres.

Example 2

In the previous example the golfer is hitting the ball to the centre of a horizontal green, level with her feet, and 75 metres away from her. Between her and the centre of the green 60 metres away is a tree of height 7 metres. Will the ball clear the tree?

Solution

In the previous example we found that $\underline{\boldsymbol{r}} = 25\boldsymbol{t}\underline{\boldsymbol{i}} + (15\boldsymbol{t} - \frac{1}{2}\boldsymbol{g}\boldsymbol{t}^2)\underline{\boldsymbol{j}} = 25\boldsymbol{t}\underline{\boldsymbol{i}} + 5\boldsymbol{t}(3 - \boldsymbol{t})\underline{\boldsymbol{j}}$

The golf ball has travelled 60 metres horizontally when $25\boldsymbol{t} = 60$ i.e. when $\boldsymbol{t} = 2{\cdot}4$ seconds.

When $\boldsymbol{t} = 2{\cdot}4$ the vertical height $= 5 \times 2{\cdot}4 \times (3 - 2{\cdot}4) = 7{\cdot}2$ metres

The golf ball will clear the tree.

Example 3

A particle is projected with initial speed 20 ms-1. Find the two possible angles of projection to achieve a range of 20 metres.

Solution

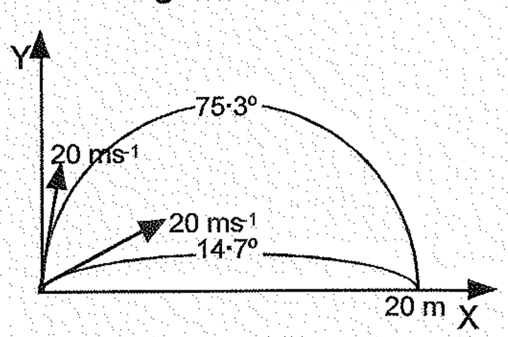
Using the formula for range, $\boldsymbol{R} = \dfrac{\boldsymbol{u}^2 \sin 2\alpha^0}{\boldsymbol{g}}$ where $\boldsymbol{u} = 20$, $\boldsymbol{R} = 20$ and $\boldsymbol{g} = 9\cdot 8$

then $20 = \dfrac{20^2 \times \sin 2\alpha^0}{9\cdot 8}$ $\quad 0 < \alpha < 90$

which gives $\sin 2\alpha^0 = \dfrac{9\cdot 8}{20} = 0\cdot 49$

so $2\alpha^0 = 29\cdot 34\ldots$ or $180 - 29.34\ldots$

$\alpha^0 \approx 14\cdot 7^0$ or $75\cdot 3^0$



The two possible paths are shown in the diagram opposite.

Example 4

A particle is projected horizontally from a raised position with a velocity of 40 ms-1. Find the horizontal and vertical components of the velocity of the particle 3 seconds after projection. Find, also, the speed and direction of motion of the particle at this time.

Solution

Starting with $\underset{\sim}{\ddot{r}} = -\boldsymbol{g}\,\underset{\sim}{j}$ and integrating

gives $\underset{\sim}{\dot{r}} = -\boldsymbol{gt}\,\underset{\sim}{j} + \underset{\sim}{c}_1$

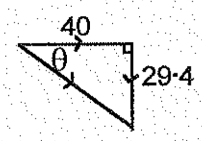
When $\boldsymbol{t} = 0$ $\quad \underset{\sim}{\dot{r}} = 40\underset{\sim}{i}$ so $\underset{\sim}{c}_1 = 40\underset{\sim}{i}$

giving $\underset{\sim}{\dot{r}} = 40\underset{\sim}{i} - \boldsymbol{gt}\,\underset{\sim}{j}$

When $\boldsymbol{t} = 3$ $\quad \underset{\sim}{\dot{r}} = 40\underset{\sim}{i} - 9\cdot 8 \times 3\underset{\sim}{j} = 40\underset{\sim}{i} - 29\cdot 4\underset{\sim}{j}$

So horizontal component $= 40$ ms^{-1} and vertical component $= -29\cdot 4$ ms^{-1}

The speed $= |\underset{\sim}{\dot{r}}| = \sqrt{40^2 + (29\cdot 4)^2} \approx 49\cdot 6$ ms^{-1}



From the velocity diagram $\tan\theta^0 = \dfrac{29\cdot 4}{40} = 0\cdot 735$

so $\theta^0 \approx 36\cdot 3^0$

At $\boldsymbol{t} = 3$ s the speed of the particle is $49\cdot 6$ ms^{-1} at an angle $36\cdot 3^0$ below the horizontal.

Exercise M1.3-1

The results obtained on the last four pages may be used without proof to answer questions 1 to 6. Take the value of g = 9·8 ms^{-2}

1. A particle is projected from a point on a horizontal plane and has an initial velocity of 140 ms^{-1} an angle of elevation of 30°. Find the greatest height reached by the particle above the plane.
2. A particle is projected from a point on a horizontal plane and has an initial velocity of 70 ms^{-1} an angle of elevation of 10°. Find the range of the particle on the horizontal plane.
3. A particle is projected from a point on a horizontal plane and has an initial velocity of $49\sqrt{2}$ ms^{-1} an angle of elevation of 45°. If the particle returns to the plane, find:
   a) the time of the flight
   b) the time taken to reach the highest point.
4. A bullet fired from a gun has a maximum horizontal range of 2000 metres. Find the muzzle velocity of the gun.
5. A gun is fired on the same horizontal level as a target 484 metres away. A bullet is fired from the gun with a muzzle velocity of 154 ms^{-1}. At what angle of elevation should the muzzle of the gun be set in order to ensure the bullet hits the target.
6. A particle projected from a point on a horizontal reaches a greatest height *h* above the plane and has a horizontal range *R*. If $R = 2h$, find the angle of projection.

In the remaining questions in this exercise establish any results which you use.

7. At time $t = 0$ a particle is projected with velocity $5\underline{i}$ ms^{-1} from a point with position vector $20\underline{j}$ m. Find the position vector of the particle when $t = 2$ seconds.
8. A stone is thrown horizontally at 21 ms^{-1} from the edge of a vertical cliff and falls to the sea, 40 metres below. Find the horizontal distance from the foot of the cliff to the point where the stone hits the sea.
9. A tennis ball is served horizontally with a speed of 21 ms^{-1} from a height of 2·8 metres. By what distance does the ball clear the a net 1 metre high situated 12 metres horizontally from the server?
10. When an aircraft is flying horizontally at a speed of 420 kmh^{-1}, it releases a bomb which, on release has the same velocity as that of the aircraft. The bomb is released when the aircraft is a distance of 2 kilometres horizontally and *h* kilometres vertically from the target. Given that the bomb hits the target, find the value of *h*.
11. Initially a particle is at the origin O and is projected with a velocity $a\underline{i}$ ms^{-1}. After *t* seconds, the particle is at the point with position vector $(30\underline{i} - 10\underline{j})$ metres. Find the value of *t* and *a*.
12. A window in a house is situated 4·9 metres above ground level. When a boy throws a ball horizontally from this window with a speed of 14 ms^{-1}, the ball just clears a vertical wall situated 10 metres from the house. Find the height of the wall and how far beyond the wall the ball first hits the ground.

Answers

1. 250 m	2. 171·0 m	3. 10 s; 5 s	4. 140 ms^{-1}
5. 5·8° or 84·2°	6. ~~60°~~ 63.4°	7. $(10\underline{i} + 0{\cdot}4\underline{j})$ m	8. 60 m
9. 0·2 m	10. 1·44	11. $\frac{10}{7}$; 21	12. 2·4 m; 4 m

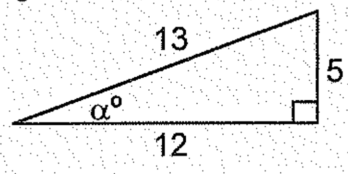
Example 5

A golfer standing on level ground hits a ball with a velocity of 52 ms-1 at an angle α° above the horizontal. If $tan\,\alpha^0 = \frac{5}{12}$ find the time for which the ball is at least 15 metres above the ground. Take $\boldsymbol{g} = 10$ ms-2

Solution

Since $0 < \alpha < 90$, and $tan\,\alpha^0 = \frac{5}{12}$, then from triangle using Pythagoras and SOH CAH TOA

$sin\,\alpha^0 = \frac{5}{13}$ and $cos\,\alpha^0 = \frac{12}{13}$.



In this problem we are only concerned when the $\boldsymbol{y}$ component of $\underline{r}$ is greater than or equal to 15 metres.

Starting with $\ddot{\boldsymbol{y}} = -\boldsymbol{g}$

Integrating gives $\dot{\boldsymbol{y}} = -\boldsymbol{gt} + \boldsymbol{c}_1$

When $\boldsymbol{t} = 0$ $\dot{\boldsymbol{y}} = 52\,sin\,\alpha^0 = 52 \times \frac{5}{13} = 20$ ms^{-1}

giving $\dot{\boldsymbol{y}} = 20 - \boldsymbol{gt}$

Integrating gives $\boldsymbol{y} = 20\boldsymbol{t} - \frac{1}{2}\boldsymbol{gt}^2 + \boldsymbol{c}_2$

When $\boldsymbol{t} = 0$ $\boldsymbol{y} = 0$ so $\boldsymbol{c}_2 = 0$

giving $\boldsymbol{y} = 20\boldsymbol{t} - \frac{1}{2}\boldsymbol{gt}^2$

When the $\boldsymbol{y}$ component $= 15$

$$15 = 20\boldsymbol{t} - \frac{1}{2} \times 10 \times \boldsymbol{t}^2$$

$$\text{so } 5\boldsymbol{t}^2 - 20\boldsymbol{t} + 15 = 0$$

$$\text{and } \boldsymbol{t}^2 - 4\boldsymbol{t} + 3 = 0$$

$$(\boldsymbol{t} - 1)(\boldsymbol{t} - 3) = 0$$

Giving the golf ball is at a height of 15 metres at $\boldsymbol{t} = 1$ and 3 s.

So the ball is at least 15 metres above the ground for $3 - 1 = 2$ s.

Example 6

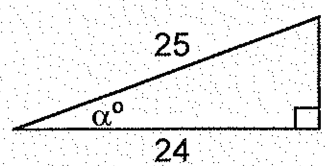
A golf ball is projected from the edge of a vertical cliff with a velocity of 50 ms-1 at angle of $sin^{-1}\frac{7}{25}$ above the horizontal. The ball strikes the sea at a point 240 metres from the foot of the cliff. Find the time for which the ball is in the air and the height of the cliff.

Solution

Let α^0 be the angle of projection.

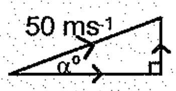
Since $0 < \alpha < 90$, and $\alpha^0 = sin^{-1}\frac{7}{25}$ then from triangle using Pythagoras and SOH CAH TOA

$cos\,\alpha^0 = \frac{24}{25}$.



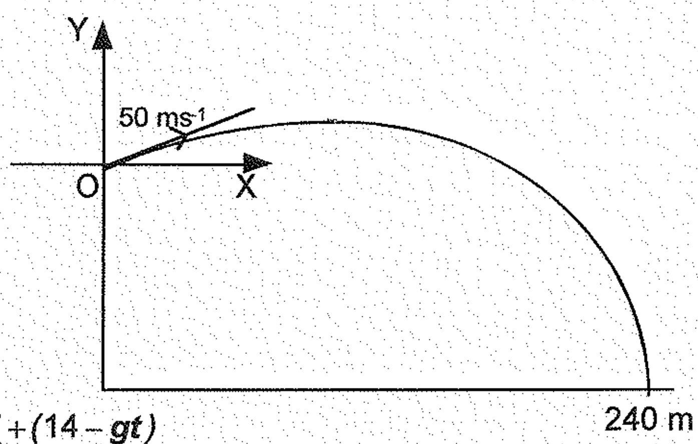
In this example we need to find a general expression for the position vector of the ball.

Starting with $\ddot{\underline{r}} = -\boldsymbol{g}\,\underline{j}$



Integrating gives $\dot{\underline{r}} = -\boldsymbol{gt}\,\underline{j} + \underline{c}_1$

When $\boldsymbol{t} = 0$ $\dot{\underline{r}} = 50\,cos\,\alpha^0\,\underline{i} + 50\,sin\,\alpha^0\,\underline{j} = 50 \times \frac{24}{25}\underline{i} + 50 \times \frac{7}{25}\underline{j} = 48\underline{i} + 14\underline{j}$



giving $\dot{\underline{r}} = 48\underline{i} + (14 - gt)$

Integrating gives $\underline{r} = 48t\underline{i} + (14t - \frac{1}{2}gt^2) + \underline{c}_2$

When $t = 0$ $\underline{r} = \underline{0}$ so $\underline{c}_2 = \underline{0}$

giving $\underline{r} = 48t\underline{i} + (14t - \frac{1}{2}gt^2)\underline{j}$

When the ball strikes the sea the x component of $\underline{r} = 240$

so $48t = 240$

Strikes sea after $\frac{240}{48} = 5$ s

The y component of $\underline{r}$ after 5 s $= 14 \times 5 - \frac{1}{2} \times 9 \cdot 8 \times 5^2 = -52 \cdot 5$

The height of the cliff $= 52 \cdot 5$ m

Exercise M1.3-2

In this exercise establish any results which you use and take $g = 9{\cdot}8$ ms^{-2}

1. A golfer hits a golf ball with a velocity of 44·1 ms^{-1} at an angle of $sin^{-1}\frac{3}{5}$ above the horizontal. The ball lands on the green at a point which is level with the point of projection. Find the time for which the ball is in the air.
2. A gun has its barrel set at an angle of elevation of 15°. The gun fires a shell with an initial speed of 210 ms^{-1}. Find the horizontal range of the shell.
3. In a game of indoor football, a free kick is awarded whenever the ball is kicked above shoulder height. The ball is initially at rest on the floor when a player kicks it with a velocity of 14 ms^{-1} at an angle of elevation θ, with $sin\,\theta = \frac{2}{5}$. Taking shoulder height as 1·5 metres and assuming nothing prevents the ball from reaching its highest point, find whether a free kick is awarded or not.
4. A ball is projected from horizontal ground and has an initial velocity of 20 ms^{-1} at an angle of elevation of $tan^{-1}\frac{7}{24}$. When the ball is travelling horizontally, it strikes a vertical wall. How high above ground level does the impact occur.
5. A particle is projected from the origin O and has an initial velocity of $30\sqrt{2}$ ms^{-1} at an angle of 45° above the horizontal. Find the horizontal and vertical displacements from O of the particle 2 seconds after projection and hence find its distance from O at that time.
6. A stone is thrown from the edge of a vertical cliff and has an initial velocity of 26 ms^{-1} at an angle $tan^{-1}\frac{5}{12}$ below the horizontal. The stone hits the sea at a point level with the base of the cliff and 72 metres from it. Find the height of the cliff and the time for which the stone is in the air. **[Take g =10 ms^{-2}]**
7. A batsman hits a ball at a velocity of 17·5 ms^{-1} angled at $tan^{-1}\frac{3}{4}$ above the horizontal, the ball initially being 0·6 metres above level ground. The ball is caught by a fielder standing 28 metres from the batsman. Find the time taken for the ball to reach the fielder and the height above the ground at which he takes the catch.
8. Ten seconds after its projection from the origin a particle has a position vector $(150\underline{i} - 200\underline{j})$ m. Find in vector form, the velocity of projection.
9. A football is kicked from a point on level ground, 15 metres from a vertical wall. Three seconds later the football hits the wall at a point 6 metres above the ground. Find the horizontal and vertical components of the initial velocity of the ball. **[Take g =10 ms^{-2}]**
10. A football is kicked from a point O on level ground and, 2 seconds later, just clears a vertical wall of height 2·4 metres. If O is 22 metres from the wall, find the velocity with which the ball is kicked.
11. A golfer hits a golf ball with a velocity of 30 ms^{-1} at an angle of $tan^{-1}\frac{4}{3}$ above the horizontal The ball lands on a green, 5 metres below the level from which it was struck. Find the horizontal distance travelled by the ball. **[Take g =10 ms^{-2}]**

Answers

1. 5·4 s	2. 2·25 km	3. Free kick	4. 1·6 m
5. 60 m; 40·4 m; 72·3 m	6. 75 m; 3 s	7. 2 s; 2 m	8. $(15\underline{i} + 29\underline{j})$ ms^{-1}
9. 5 ms^{-1}; 17 ms^{-1}	10. 15·6 ms^{-1} at 45° to horizontal		11. 90 m

Example 6

A particle is projected from the origin O with initial speed 20 ms^{-1} to pass through a point at 10 metres from O horizontally and 10 metres above O. Show that there there are two possible angles of projection and find the corresponding ranges. [Take g =10 ms^{-2}]

Solution

Let the angle of projection be α^0.

Since we know a point on the trajectory and the speed of projection will use the Cartesian equation established earlier.

$$y = x\tan\alpha^0 - \frac{g}{2u^2}(1+\tan^2\alpha^0)x^2 \text{ where } x = y = 10,\ g = 10 \text{ and } u = 20.$$

$$10 = 10\tan\alpha^0 - \frac{10}{2\times 20^2}\times(1+\tan^2\alpha^0)\times 10^2$$

$$10 = 10\tan\alpha^0 - 1{\cdot}25\times(1+\tan^2\alpha^0)$$

$$1{\cdot}25\tan^2\alpha^0 - 10\tan\alpha^0 + 11{\cdot}25 = 0$$

Mutiplying through by 4 gives:

$$5\tan^2\alpha^0 - 40\tan\alpha^0 + 45 = 0$$

Dividing through by 5 gives:

$$\tan^2\alpha^0 - 8\tan\alpha^0 + 9 = 0 \text{ which cannot be factorised.}$$

Using the quadratic formula, $\tan\alpha^0 = \dfrac{8\pm\sqrt{(-8)^2 - 4\times 1\times 9}}{2\times 1}$

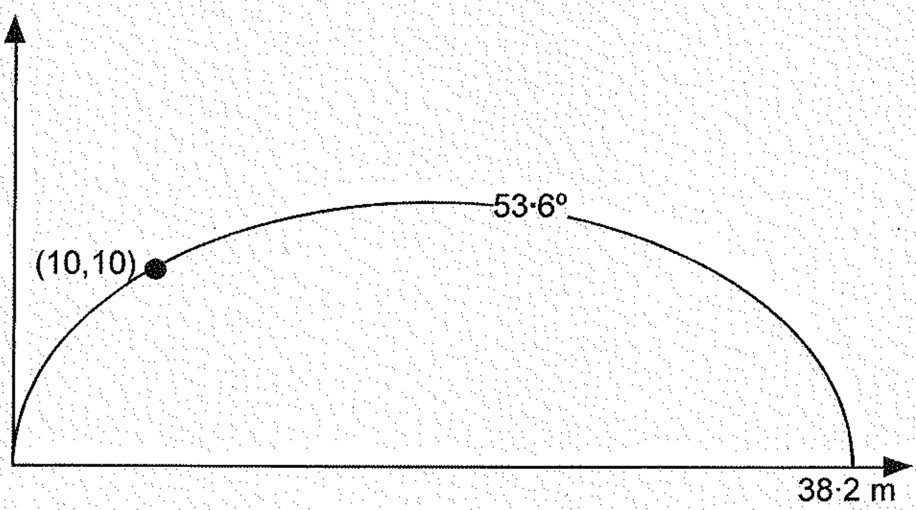
$$\tan\alpha^0 = 6{\cdot}645\ldots \text{ or } 1{\cdot}354\ldots$$

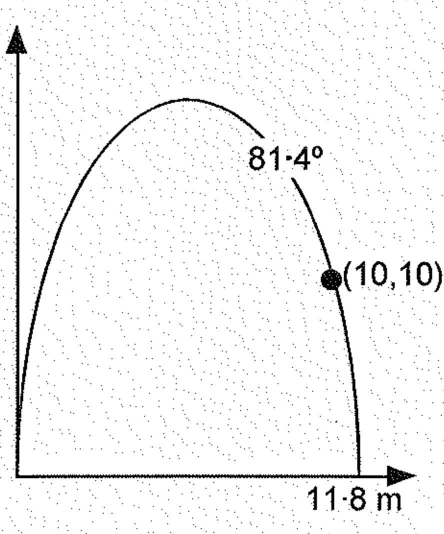
$$\text{giving } \alpha^0 \approx 81{\cdot}4^0 \text{ or } 53{\cdot}6^0$$

Using formula for range $R = \dfrac{u^2\sin 2\alpha^0}{g}$

When $\alpha^0 = 53{\cdot}6^0$, $R = \dfrac{20^2\sin(2\times 53{\cdot}6^0)}{10} \approx 38{\cdot}2$ m and when $\alpha^0 = 81{\cdot}4^0$, $R = \dfrac{20^2\sin(2\times 81{\cdot}4^0)}{10} \approx 11{\cdot}8$ m

The corresponding trajectories are sketched below for completeness.





Exercise M1.3-3

The equation of the trajectory obtained on page 36 may be used without proof to answer questions 1 to 5. Take the value of $g = 9{\cdot}8$ ms^{-2} unless told otherwise.

1. A particle is projected from a point on a horizontal plane and initially travels in a direction which makes an angle of $\tan^{-1}\frac{3}{4}$ with the horizontal. In the subsequent motion the particle passes through a point above the plane which is 20 metres horizontally and 10 metres vertically from the point of projection. Find the speed of projection. **[Take g = 10 ms^{-2}]**
2. A particle is projected from a point on a horizontal plane and has an initial speed of 28 ms^{-1}. If the particle passes through a point above the plane, 40 metres horizontally and 20 metres vertically from the point of projection, find the possible angles of projection.
3. A particle is projected from a point on a horizontal plane and has an initial speed of 42 ms^{-1}. If the particle passes through a point above the plane, 60 metres horizontally and 70 metres vertically from the point of projection, find the possible angles of projection.
4. The origin O lies on a horizontal plane and a point P lies above the plane, 50 metres horizontally and 60 metres vertically from O. Show that a particle projected from O with a speed 35 ms^{-1} cannot pass through P.
5. A ball is thrown from ground level with speed 40 ms^{-1}. It just clears a wall 30 metres high, 40 metres from the point of projection. Show that there are two possible angles of projection, one of which is 45°. Find in each case:
   (i) the greatest height
   (ii) the range
   reached on each trajectory. **[Take g = 10 ms^{-2}]**

In the remaining questions in this exercise establish any results that you use.

6. A particle is projected with initial speed 70 ms^{-1} from the top of a cliff of height 40 metres, and falls into the sea at a distance 200 metres from the bottom of the cliff. Find the two possible angles of projection and the difference between the corresponding times of flight.
7. (i) A particle is projected with initial speed u to pass through the point (x,y). Show that, if α° is the angle of projection:

$$\frac{gx^2}{2u^2}\tan^2\alpha^0 - x\tan\alpha^0 + \left(\frac{gx^2}{2u^2} + y\right) = 0$$

   Hence show that if the point (x,y) can just be reached with initial speed u, then

$$y = -\frac{g}{2u^2}x^2 + \frac{u^2}{2g}$$

   [Hint: think of the first equation as a quadratic in $\tan\alpha^o$; what can you say about its roots if (x,y) can ***just*** be reached?]

   Sketch this curve. It is called the **parabola of safety**.

   (ii) A golfer can hit a ball at 20 ms^{-1}. There is a tree 20 metres in front of him. Show that he can clear the tree if it is less than 15 metres high. **[Take g = 10 ms^{-2}]**

Answers

1. 25 ms^{-1} 2. 45°; 71·6° 3. 63·4°; 76·0°
5. (i) 40 m, 78·4 m (ii) 160 m, 44·8 m 6. 0°, 78·7°; 11·7 s

M1.4 FORCE AND NEWTON'S LAWS OF MOTION

Outcome Content
Understand the terms mass, force, momentum, balanced and unbalanced forces, resultant force, equilibrium, resistive forces.
Know Newton's first and third laws of motion.
Resolve forces in two dimensions to find their components.
Combine forces to find the resultant force.
Understand the concept of static and dynamic friction and limiting friction.
Understand the terms frictional force, normal reaction, coefficient of friction μ, angle of friction λ and know the equations $F = \mu R$ and $\mu = \tan\lambda$.
Solve problems involving a particle or body in equilibrium under the action of certain forces.
Solve problems involving friction and problems on inclined planes.

Equilibrium

If a particle is at rest or moving with constant velocity it is said to be in **equilibrium**.
By **Newton's First Law of Motion**, a body is in equilibrium if the net force acting on it is zero, i.e. the system of forces acting on the body is **balanced**.

A typical problem in this section of the Unit involves finding the magnitude of some force within a system of balanced forces acting on a body.

The method that will be used in this unit is to:

i) resolve the forces in two perpendicular directions

ii) show that the components in each of these directions 'balance out'.

Example

A mass of 8 kg hangs in equilibrium, suspended by two light, inelastic strings making angles of 30° and 45° with the horizontal, as shown in the diagram below on the right. Calculate the tensions in the two strings.

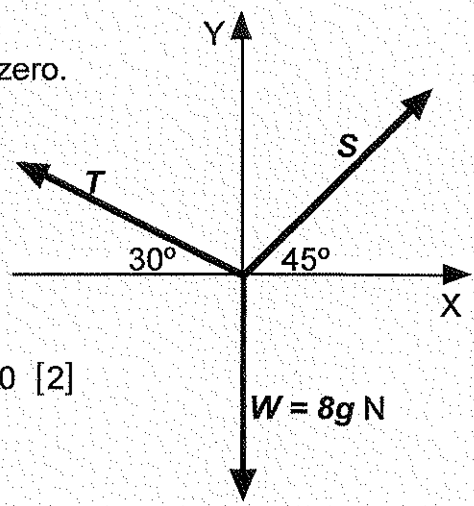
Solution

The mass is hanging in equilibrium, so the net force acting is zero.

This can be written as the vector equation $\sum \underline{F} = \underline{0}$

As suggested above, by resolving the forces, the vector equation can be written as two scalar equations.

$$\sum F_X = 0 \text{ and } \sum F_Y = 0$$



Resolving gives

$$S\cos 45^0 - T\cos 30^0 = 0 \; [1] \quad \text{and} \quad S\sin 45^0 + T\sin 30^0 - W = 0 \; [2]$$

From [1] $S = \dfrac{T\cos 30^0}{\cos 45^0}$ and substituting for S in [2] gives

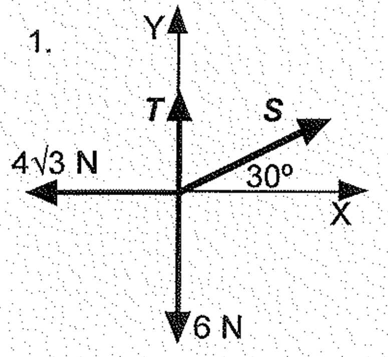
$$\frac{T\cos 30^0}{\cos 45^0} \times \sin 45^0 + T\sin 30^0 = 8g$$

Giving $T = \dfrac{8g}{\cos 30^0 \tan 45^0 + \sin 30^0} \approx 57 \cdot 4$ N and $S = \dfrac{57 \cdot 4 \times \cos 30^0}{\cos 45^0} \approx 70 \cdot 3$ N

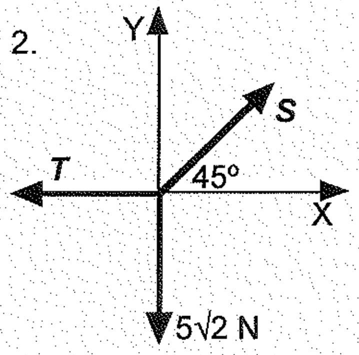
Exercise M1.4-1

Each of the diagrams in questions 1 to 6 shows a particle in equilibrium under the forces shown. In each case find the unknown forces and angles.

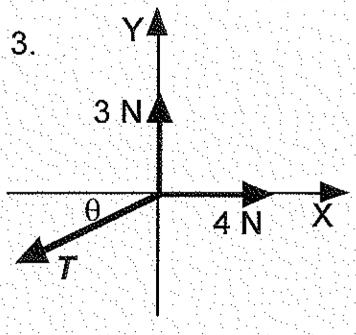
1.



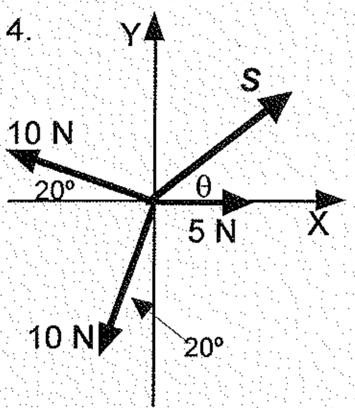
2.



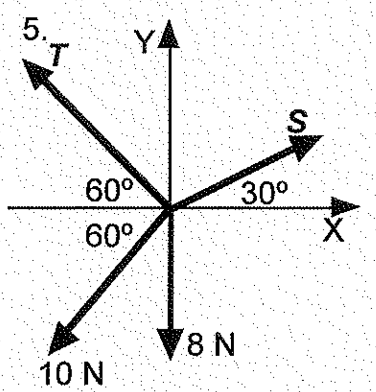
3.



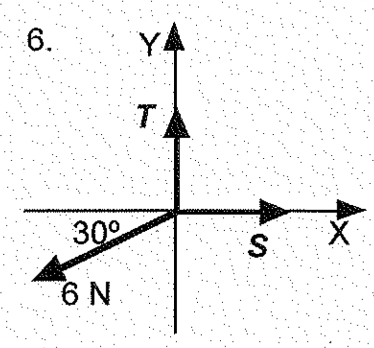
4.



5.



6.



Answers

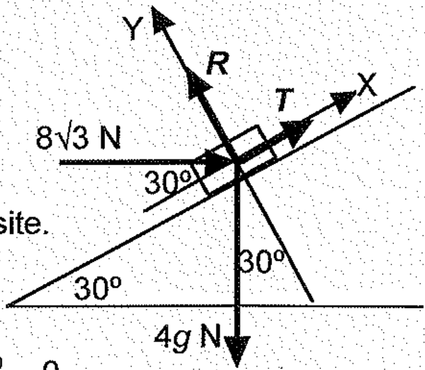
1. $T = 2$ N; $S = 8$ N 2. $T = 5\sqrt{2}$ N; $S = 10$ N 3. $T = 5$ N; $\theta \approx 36{\cdot}9°$ 4. $T \approx 9{\cdot}84$ N; $\theta \approx 37{\cdot}4°$
5. $S = (5\sqrt{3} + 4)$ N $\approx 12{\cdot}7$ N; $T = (5 + 4\sqrt{3})$ N $\approx 11{\cdot}9$ N 6. $S = 3\sqrt{3}$ N; $T = 3$ N

Resolving in other directions

Resolving in a horizontal and a vertical direction is often found convenient, but this is not always the best choice. For example, on an inclined plane, then it will usually found that a better choice is to resolve the forces parallel to, and at right angles to, the surface of the plane.

Example

A particle of mass 4 kg rests on the surface of a smooth plane which is inclined at an angle of 30° to the horizontal. When a force T acting up the plane and a horizontal force of $8\sqrt{3}$ N are applied to the particle, it rests in equilibrium. Calculate T and the normal reaction between the particle and the plane.



Solution

The particle is in equilibrium, so the net force acting is zero.

$$\sum F_X = 0 \text{ and } \sum F_Y = 0$$

Let the normal reaction be R as shown in the force diagram opposite.

This diagram is also called a free body diagram.

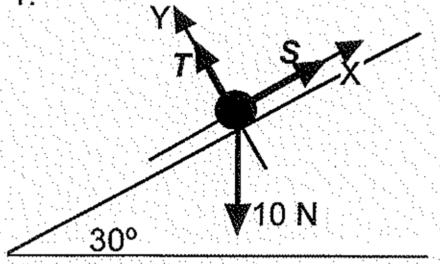
Resolving gives

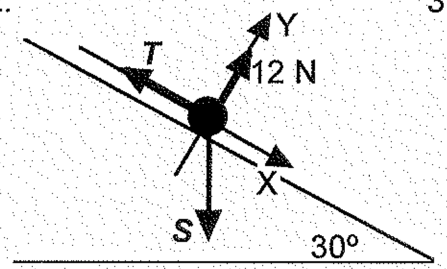
$$T + 8\sqrt{3}\cos 30^0 - 4g\sin 30^0 = 0 \text{ and } R - 8\sqrt{3}\sin 30^0 - 4g\cos 30^0 = 0$$

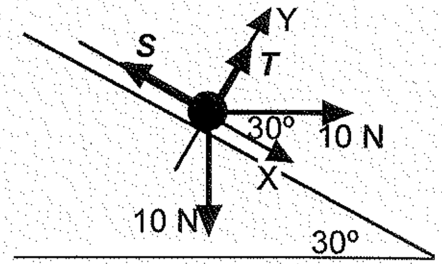
Giving $T = 4g\sin 30^0 - 8\sqrt{3}\cos 30^0 = 7\cdot 6$ N and $R = 4g\cos 30^0 + 8\sqrt{3}\sin 30^0 \approx 40\cdot 9$ N

Exercise M1.4-2

Each of the diagrams in questions 1 to 4 shows a particle in equilibrium under the forces shown. In each case, using the axes given, find the unknown forces and angles.

1.


2.


3.


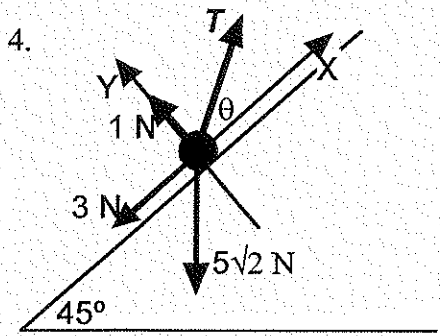
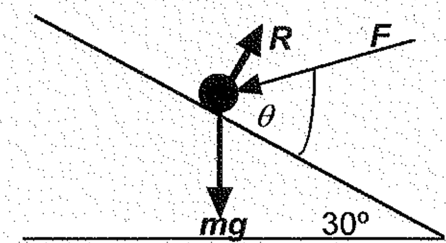
4.


Diagram for question 7


5. A force acting parallel to and up the line of greatest slope holds a body of mass 10 kg in equilibrium on a smooth plane which is inclined at 30° to the horizontal. Find the magnitude of this force and of the normal reaction between the body and the plane.
6. A horizontal force P holds a body of mass 10 kg in equilibrium on a smooth plane which is inclined at 30° to the horizontal. Find the magnitude of P and of the normal reaction between the body and the plane.
7. A force F holds a particle of mass m in equilibrium on a smooth plane which is inclined at 30° to the horizontal. If F makes an angle θ with the plane, as shown in the diagram above, find θ when R, the normal reaction between particle and plane, is $1{\cdot}5mg$.
8. A particle of mass 3 kg lying on a smooth surface which is inclined at $\theta°$ to the horizontal is attached to a light inextensible string which passes up the plane, along the line of greatest slope, over a frictionless pulley at the top and carries a 1 kg mass freely suspended at its end. If the system rests in equilibrium, find:
   a) the value of θ
   b) the tension in the string
   c) the normal reaction between the particle and the plane.

Answers

1. $T = 5\sqrt{3}$ N; $S = 5$ N
2. $T = 4\sqrt{3}$ N; $S = 8\sqrt{3}$ N
3. $T = 5(\sqrt{3} - 1)$ N; $S = 5(\sqrt{3} + 1)$ N
4. $T \approx 8{\cdot}94$ N; $\theta \approx 26{\cdot}6°$
5. 49 N; 84·9 N
6. 56·6 N; 113 N
7. 51·7°
8. a) 19·5° b) 9·8 N c) 27·7 N

Resultant of a System of Forces

The **resultant** force *R*, of a system of forces, is that single force which could completely take the place of the force system. The resultant force *R* must have the same effect as the system of forces.

Example 1

The resultant of the forces $(5\underline{i} + 7\underline{j})$ N, $(a\underline{i} + b\underline{j})$ N and $(b\underline{i} - a\underline{j})$ N is a force$(11\underline{i} + 5\underline{j})$ N. Find *a* and *b*.

Solution

Using $\underline{\boldsymbol{R}} = \sum \underline{\boldsymbol{F}}$ then $11\underline{\boldsymbol{i}} + 5\underline{\boldsymbol{j}} = 5\underline{\boldsymbol{i}} + 7\underline{\boldsymbol{j}} + \boldsymbol{a}\underline{\boldsymbol{i}} + \boldsymbol{b}\underline{\boldsymbol{j}} + \boldsymbol{b}\underline{\boldsymbol{i}} - \boldsymbol{a}\underline{\boldsymbol{j}}$

which simplifies to $11\underline{\boldsymbol{i}} + 5\underline{\boldsymbol{j}} = (5 + \boldsymbol{a} + \boldsymbol{b})\underline{\boldsymbol{i}} + (7 + \boldsymbol{b} - \boldsymbol{a})\underline{\boldsymbol{j}}$

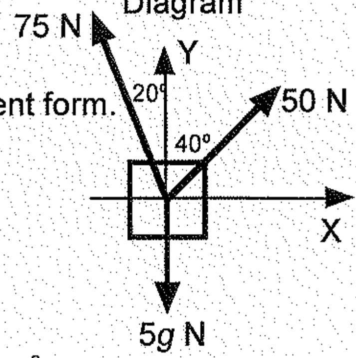
So $11 = 5 + \boldsymbol{a} + \boldsymbol{b}$ giving $\boldsymbol{a} + \boldsymbol{b} = 6$ and $7 + \boldsymbol{b} - \boldsymbol{a} = 5$ giving $\boldsymbol{b} - \boldsymbol{a} = -2$

Adding the two equations gives $2\boldsymbol{b} = 4$ so $\boldsymbol{b} = 2$ and $\boldsymbol{a} = 4$

Example 2

A body of mass 5 Kg is being raised by two forces of 75 N and 50 N as shown in the diagram below. Find, the magnitude of the resultant force acting on the body, and find the angle this resultant makes with the upward vertical.

Free Body Diagram



Solution

From the definition of a resultant force, the resultant $\underline{\boldsymbol{R}} = \sum \underline{\boldsymbol{F}}$.

To carry out this summation the forces must be expressed in component form.

In component form the weight $5\boldsymbol{g}$ N $= -5\boldsymbol{g}\,\underline{\boldsymbol{j}}$ N

and the 50 N force $= 50\cos 50^0\,\underline{\boldsymbol{i}} + 50\sin 50^0\,\underline{\boldsymbol{j}}$ N

and the 75 N force $= -75\cos 70^0\,\underline{\boldsymbol{i}} + 75\sin 70^0\,\underline{\boldsymbol{j}}$ N

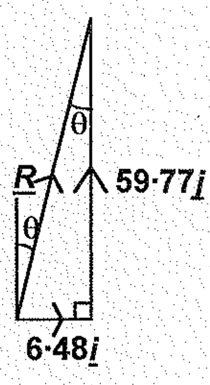
This gives

$$\underline{\boldsymbol{R}} = \sum \underline{\boldsymbol{F}} = -5\boldsymbol{g}\,\underline{\boldsymbol{j}} + 50\cos 50^0\,\underline{\boldsymbol{i}} + 50\sin 50^0\,\underline{\boldsymbol{j}} + (-75\cos 70^0\,\underline{\boldsymbol{i}} + 75\sin 70^0\,\underline{\boldsymbol{j}})$$

so $\underline{\boldsymbol{R}} = 6\cdot 48\ldots\underline{\boldsymbol{i}} + 59\cdot 77\ldots\underline{\boldsymbol{j}}$ N

For the magnitude of the resultant, $|\underline{\boldsymbol{R}}|^2 = (6\cdot 48\ldots)^2 + (59\cdot 77\ldots)^2 = 3615\cdot 6\ldots$

so $|\underline{\boldsymbol{R}}| \approx 60\cdot 1$ N



From the components of $\underline{\boldsymbol{R}}$ the angle made with the vertical can be found.

$$\tan\theta^0 = \frac{6\cdot 48}{59\cdot 77} = 0\cdot 108\ldots$$

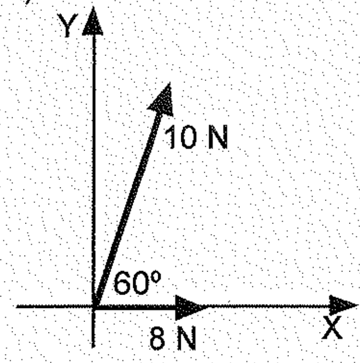
So $\theta^0 \approx 6\cdot 1^0$

As the body is not constrained to move, say vertically it will follow the path given by the direction of $\underline{\boldsymbol{R}}$. This is $6\cdot 1^0$ from the vertical to the right.

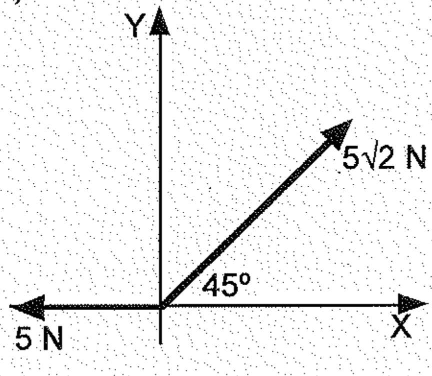
Exercise 1.4-3

1. The resultant of the forces $(5\underline{i} - 2\underline{j})$ N, $(7\underline{i} + 4\underline{j})$ N, $(a\underline{i} + b\underline{j})$ N and $(-3\underline{i} + 2\underline{j})$ N is a force $(5\underline{i} + 5\underline{j})$ N. Find a and b.
2. Find the magnitude of the force $(-2\underline{i} + 4\underline{j})$ N and the angle it makes with the direction of $\underline{i}$.
3. For each of the following systems of forces find the resultant in the form $a\underline{i} + b\underline{j}$. Hence find the magnitude of the resultant and the angle it makes with the the X-axis.

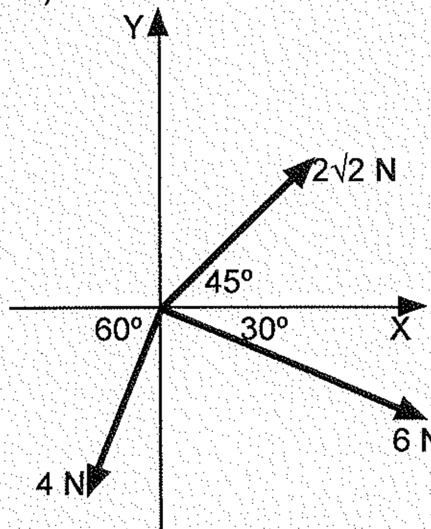
a)



b)

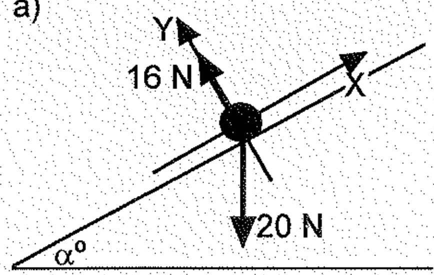


c)



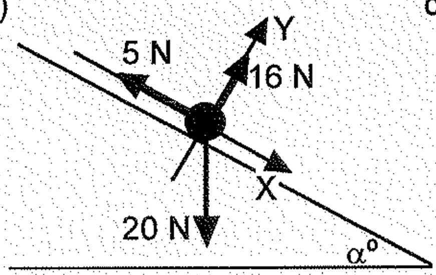
4. Find the magnitude and direction of the resultant of each of the following systems of forces.

a)



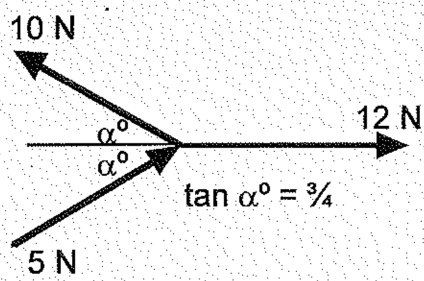
tan α° = ¾

b)



tan α° = ¾

c)



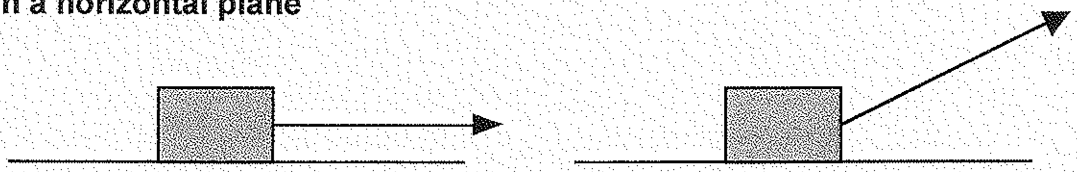
5. A sledge is being pulled across a horizontal surface by forces of $(6\underline{i} + 2\underline{j})$ N and $(4\underline{i} - 3\underline{j})$ N. What is the magnitude of the resultant pull on the sledge and what angle does this resultant make with the direction of $\underline{i}$?
6. Find the magnitude and direction of the resultant of forces of 10 N, 15 N and 8 N acting in the directions 030°, 150° and 225° respectively.

Answers

1. $a = -4$; $b = 1$
2. 4·47 N; 116·6° ac/w from $\underline{i}$
3. a) $(13\underline{i} + 8{\cdot}66\underline{j})$ N; 15·6 N; 33·7° ac/w
   b) $5\underline{j}$ N; 5 N; 90° ac/w
   c) $(5{\cdot}2\underline{i} - 4{\cdot}46\underline{j})$ N; 6·85 N; 40·7° c/w
4. a) 12 N down slope.
   b) 7 N down slope.
   c) 12·04 N at 48·4° ac/w from 12 N force.
5. √101 N; 5·7° c/w from $\underline{i}$
6. 12·1 N; bearing 145·6°

PROBLEMS INVOLVING FRICTION AND BODIES AT REST

Bodies on a horizontal plane



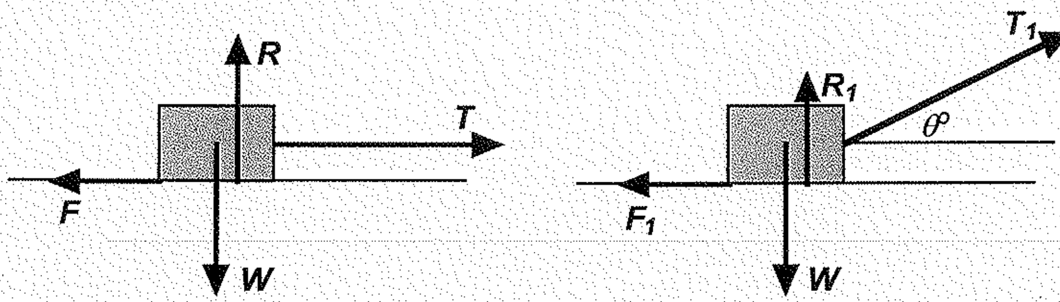
The above diagrams show a block **at rest** on a horizontal plane being "pulled" by a string which, on the left is parallel to the plane and on the right is inclined at an angle to the horizontal. There are four forces acting on the block, namely

a) the **weight** of the block, W, acting vertically downwards

b) the **normal reaction**, R, of the plane acting at right angles to the plane (The block exerts a force on the plane and the plane exerts an equal and opposite force on the block. This is an illustration of **Newton's third law**)

c) the **tension** in the string, T, acting along the direction of the string

d) a **resistive force**, F, the **friction** between the block and the plane as shown in the free body diagrams below.



The **weight** of the block, like the other three forces is measured in Newtons and $W = mg$, where m is the mass of the block in kilograms and g is the acceleration due to gravity, measured in ms^{-2}.

Friction is the name given to the force which tends to prevent slipping between surfaces in contact. Whilst there is always some force between slipping surfaces it is sometimes very small, in which case we neglect it, and say that the contact between the surfaces is **smooth**. Eg. between an ice hockey puck and the ice it moves across.
Surfaces which are not smooth are said to be **rough**. Any rough surface will involve frictional forces. Friction always opposes slipping between two surfaces in contact. This does not mean that friction always opposes motion. You would find it difficult walking to school if there was no friction between your shoes and the ground.

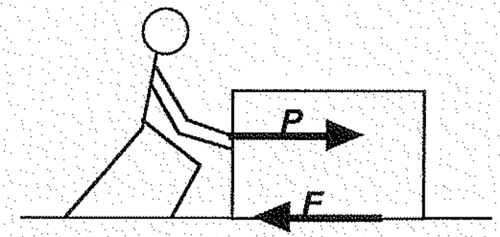
The fundamental properties of friction are:

a) Friction always opposes the relative motion of surfaces in contact and acts at a tangent to the surfaces.
b) Friction Is a variable force and when slipping does not take place it is just sufficient to prevent relative motion of the surfaces in contact.

When there is no motion, as above, we have **static friction** and when there is motion we have kinetic or **dynamic friction** and, when the block is on the point of moving we have **limiting friction**. From experimental data is has been observed that:

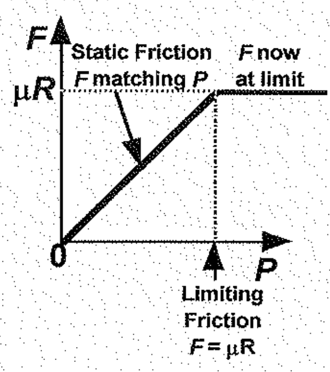
The Limiting Frictional Force $= \mu$ **x The Normal Reaction Force**

where μ is a constant, called the **coefficient of friction**, which depends on the materials of the block and the plane.



Consider the man above pushing a heavy box. The man pushes with a force P. Assuming the box does not tip, the frictional force F is equal in magnitude and opposite in direction to P. When the box does not move this is called **static friction**. When the box is on the point of moving the size of the frictional force has reached an upper limit. This situation is called **limiting equilibrium**. Even if P gets bigger F remains at this limiting value. While the box is moving, this limiting value of friction is called **dynamic friction**. As outlined in the formula above this limit depends on how rough the surfaces are and how heavy the box is.

The graph opposite illustrates how F varies as P is increased.



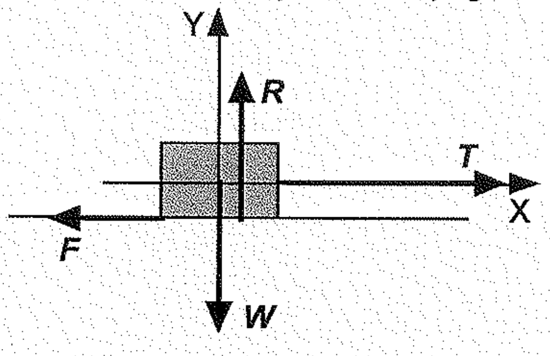
In summary, for static friction $F < \mu R$

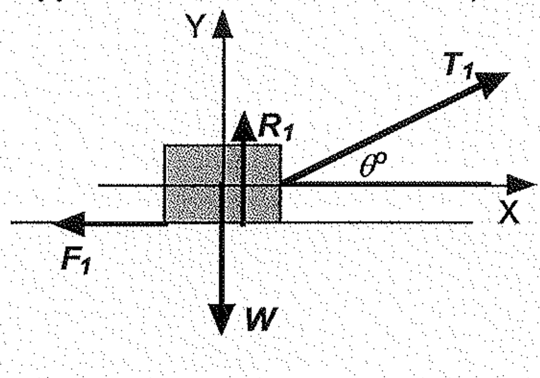
and for limiting equilibrium $F = \mu R$
or dynamic friction

Some typical values for μ are:

Surfaces	μ
Metal on ice	0·01
Metal on metal	0·1 to 0·3
Rubber on road	approaching 1·0

Returning to the two force systems on page 51. Suppose that the block is on the point of moving.





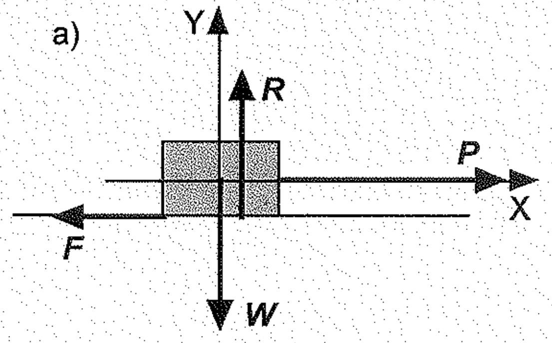
	Horizontal String	Inclined String
Limiting Equilibrium	so $\boldsymbol{F} = \mu \boldsymbol{R}$ and $\sum \underline{F} = \underline{0}$	$F_1 = \mu R_1$ and $\sum \underline{F} = \underline{0}$
	so $\sum F_X = 0$ and $\sum F_Y = 0$	so $\sum F_X = 0$ and $\sum F_Y = 0$
	Giving $T - F = 0$ and $R - W = 0$	Giving $T_1 \cos\theta^0 - F_1 = 0$ and $R_1 + T_1 \sin\theta^0 - W = 0$
	$T = F$ and $R = W$	$T_1 = \dfrac{F_1}{\cos\theta^0}$ and $R_1 = W - T_1 \sin\theta^0$
	$T = \mu R$ and $R = mg$	$T_1 = \dfrac{\mu R_1}{\cos\theta^0}$ and $R_1 = mg - T_1 \sin\theta^0$

WORKED EXAMPLES

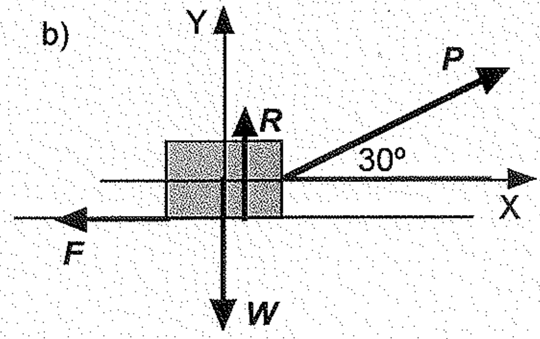
Example 1

A 5 kg block of wood is at rest on a rough horizontal surface. The coefficient of friction between the block and the surface is 0·6. Calculate the magnitude of the force ***P*** which is necessary for motion to occur if ***P*** is applied to the block a) horizontally and b) at an angle of 30° above the horizontal, as shown in the diagrams.

a)



b)



Solution

a) In limiting equilibrium so $\sum F_X = 0$ and $\sum F_Y = 0$ and $F = \mu R$

$$\text{So } P - F = 0 \text{ and } R - W = 0$$

$$\text{Giving } P = \mu R = \mu mg = 0{\cdot}6 \times 5 \times 9{\cdot}8 = 29{\cdot}4 \text{ N}$$

b) In limiting equilibrium so $\sum F_X = 0$ and $\sum F_Y = 0$ and $F = \mu R$

$$\text{So } P\cos 30^0 - F = 0 \text{ and } R + P\sin 30^0 - W = 0$$

$$\text{Giving } P = \frac{\mu R}{\cos 30^0} \text{ and } R = W - P\sin 30^0$$

$$\text{So } P = \frac{\mu(mg - P\sin 30^0)}{\cos 30^0}$$

$$\tfrac{\sqrt{3}}{2}P = 0{\cdot}6 \times 5 \times 9{\cdot}8 - 0{\cdot}6 \times \tfrac{1}{2} \times P$$

$$(\tfrac{\sqrt{3}}{2} + 0{\cdot}3)P = 29{\cdot}4$$

$$P \approx 25{\cdot}2 \text{ N}$$

So in (a) the force ***P*** must just exceed $29{\cdot}4$ N and in (b) $25{\cdot}2$ N for motion to take place.

Example 2

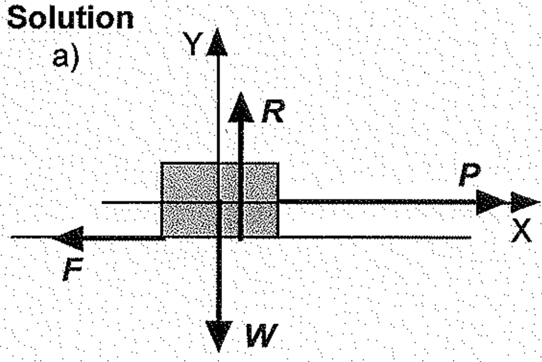
A block of mass 20 kg rests on a horizontal plane whose coefficient of friction is 0·4. Find the least force to move the block if it acts:

a) horizontally b) at 30° above the horizontal c) at 30° below the horizontal

d) at the most favourable angle.

Solution

a)

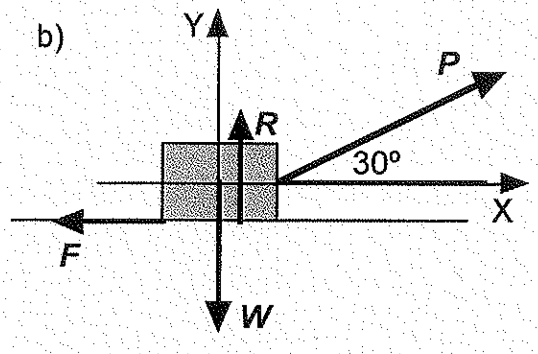


On point of moving so in limiting equilibrium

$\sum F_X = 0$ and $\sum F_Y = 0$ and $F = \mu R$

So $P - F = 0$ and $R - W = 0$

Giving $P = \mu R = \mu mg = 0{\cdot}4 \times 20 \times 9{\cdot}8 = 78{\cdot}4$ N b)



On point of moving so in limiting equilibrium

$\sum F_X = 0$ and $\sum F_Y = 0$ and $F = \mu R$

So $P\cos 30^0 - F = 0$ and $R + P\sin 30^0 - W = 0$

Giving $P = \dfrac{\mu R}{\cos 30^0}$ and $R = W - P\sin 30^0$

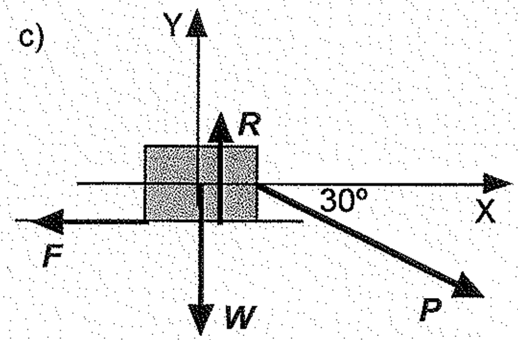
So $P = \dfrac{\mu(mg - P\sin 30^0)}{\cos 30^0}$

$$\tfrac{\sqrt{3}}{2}P = 0\cdot 4 \times 20 \times 9\cdot 8 - 0\cdot 4 \times \tfrac{1}{2} \times P$$

$$(\tfrac{\sqrt{3}}{2} + 0\cdot 2)P = 78\cdot 4$$

$$P \approx 73\cdot 5 \text{ N}$$

c)



On point of moving so in limiting equilibrium

$\sum F_X = 0$ and $\sum F_Y = 0$ and $F = \mu R$

So $P\cos 30^0 - F = 0$ and $R - P\sin 30^0 - W = 0$

Giving $P = \dfrac{\mu R}{\cos 30^0}$ and $R = W + P\sin 30^0$

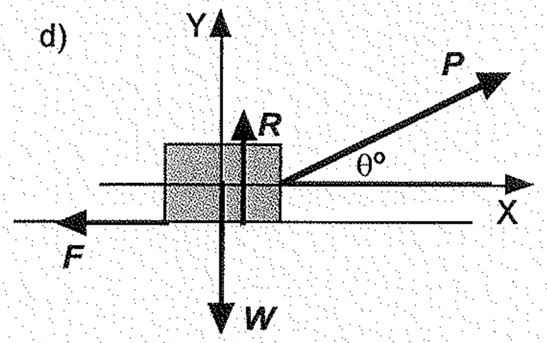
So $P = \dfrac{\mu(mg + P\sin 30^0)}{\cos 30^0}$

$$\tfrac{\sqrt{3}}{2}P = 0\cdot 4 \times 20 \times 9\cdot 8 + 0\cdot 4 \times \tfrac{1}{2} \times P$$

$$(\tfrac{\sqrt{3}}{2} - 0\cdot 2)P = 78\cdot 4$$

$$P \approx 117\cdot 7 \text{ N}$$

d)



On point of moving so in limiting equilibrium

$\sum F_X = 0$ and $\sum F_Y = 0$ and $F = \mu R$

So $P\cos\theta^0 - F = 0$ and $R + P\sin\theta^0 - W = 0$

Giving $P = \dfrac{\mu R}{\cos\theta^0}$ and $R = W - P\sin\theta^0$

So $P = \dfrac{\mu(mg - P\sin\theta^0)}{\cos\theta^0}$

$$\cos\theta^0 \times P = 0\cdot 4 \times 20 \times 9\cdot 8 - 0\cdot 4 \times \sin\theta^0 \times P$$

$$(\cos\theta^0 + 0\cdot 4\sin\theta^0)P = 78\cdot 4$$

$$P = \frac{78\cdot 4}{(\cos\theta^0 + 0\cdot 4\sin\theta^0)}$$

Note: You will not be expected to recall from Higher work how to express *acos* θ° + *bsin* θ° as *kcos(*θ − α*)*°.
Part (d) was only included for 'completeness'.

Using $k\cos(\theta - \alpha)^0 = \cos\theta^0 + 0\cdot 4\sin\theta^0$ it can be shown that $k = \sqrt{1\cdot 16}$ and $\alpha^0 \approx 21\cdot 8^0$.

$$\text{So } P = \frac{78\cdot 4}{\sqrt{1\cdot 16}\cos(\theta - 21\cdot 8)^0}$$

For P to have a minimum value $\cos(\theta - 21\cdot 8)^0$ must have its maximum value of 1.

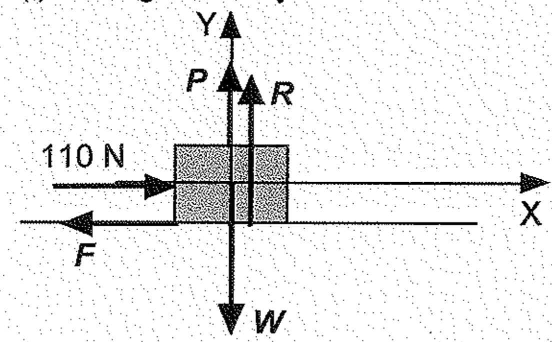
So least value of $P = \dfrac{78\cdot 4}{\sqrt{1\cdot 16}} \approx 72\cdot 8$ N

Example 3

A worker tries unsuccessfully to move a 35 kg crate, resting on a horizontal floor, by pushing the crate horizontally with a force of 110 N. A second worker decides to help. What is the minimum pulling force she would need to apply (i) vertically (ii) horizontally to help move the crate if the coefficient of friction is 0·37?

Solution

(i) Pulling vertically



On point of moving so in limiting equilibrium

$\sum F_X = 0$ and $\sum F_Y = 0$ and $\boldsymbol{F} = \mu \boldsymbol{R}$

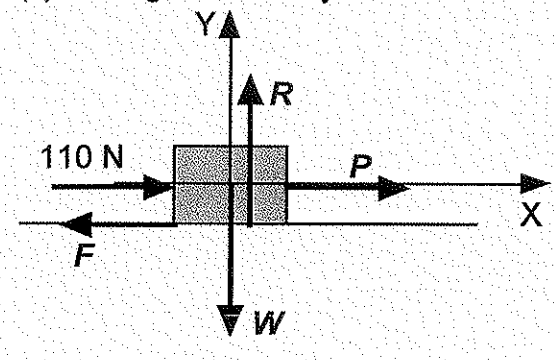
So $110 - \boldsymbol{F} = 0$ and $\boldsymbol{R} + \boldsymbol{P} - \boldsymbol{W} = 0$

Giving $110 = \mu \boldsymbol{R} = \mu(\boldsymbol{mg} - \boldsymbol{P})$

So $110 = 0 \cdot 37 \times 35 \times 9 \cdot 8 - 0 \cdot 37 \times \boldsymbol{P}$

$$\boldsymbol{P} = \frac{0 \cdot 37 \times 35 \times 9 \cdot 8 - 110}{0 \cdot 37} \approx 45 \cdot 7 \text{ N}$$

(ii) Pulling horizontally



On point of moving so in limiting equilibrium

$\sum F_X = 0$ and $\sum F_Y = 0$ and $\boldsymbol{F} = \mu \boldsymbol{R}$

So $110 + \boldsymbol{P} - \boldsymbol{F} = 0$ and $\boldsymbol{R} - \boldsymbol{W} = 0$

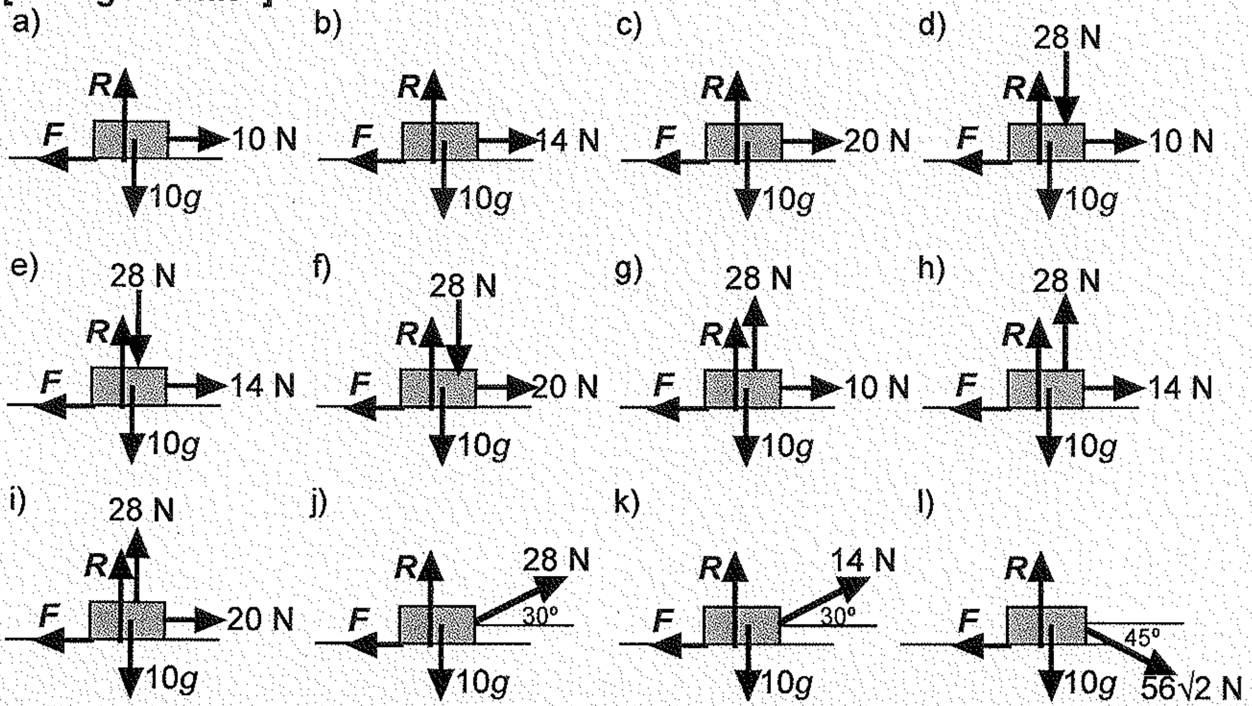
Giving $110 + \boldsymbol{P} = \mu \boldsymbol{R} = \mu \boldsymbol{mg}$

So $110 + \boldsymbol{P} = 0 \cdot 37 \times 35 \times 9 \cdot 8$

$\boldsymbol{P} = 0 \cdot 37 \times 35 \times 9 \cdot 8 - 110 = 16 \cdot 91 \text{ N}$

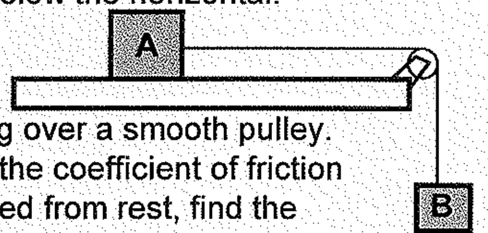
Exercise 1.4-4

1. Each of the following diagrams shows a body of mass 10 kg initially at rest on a rough horizontal plane. The coefficient of friction between the body and the plane is 0·14. All letters have the same meaning as used in the notes. In each case find the magnitude of ***F*** and state whether the body will remain at rest or will accelerate along the plane. **[Take g = 10 ms^{-2}]**



2. When a horizontal force of 28 N is applied to a body of mass 5 kg which is resting on a horizontal plane, the body is found to be in limiting equilibrium. Find the coefficient of friction between the body and the plane.
3. A block of mass 20 kg rests on a rough horizontal plane. The coefficient of friction between the block and the plane is 0·25. Calculate the frictional force experienced by the block when a horizontal force of 50 N acts on the block. State whether the block will move.
4. A box of mass 2 kg lies on a rough horizontal floor, coefficient of friction 0·2. A light string is attached to the box in order to pull the box across the floor. If the tension in the string is ***T*** N, find the value that ***T*** must exceed for motion to occur if the string is:
   a) horizontal b) 45° above the horizontal c) 45° below the horizontal.

Questions 5 and 6 refer to the system shown in the diagram Body A lies on a rough horizontal table and is connected to the freely hanging body B by a light inextensible string passing over a smooth pulley.



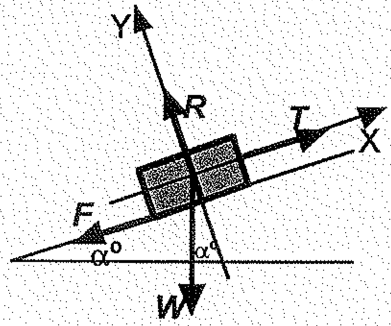
5. The masses of A and B are 6 kg and 1 kg respectively and the coefficient of friction between body A and the table is 0·2. If the system is released from rest, find the frictional force experienced by A and state whether motion will occur.
6. The masses of A and B are m_1 and m_2 respectively and the coefficient of friction between body A and the table is μ. Show that if the system is released from rest, motion will occur if $m_2 > \mu m_1$.

Answers

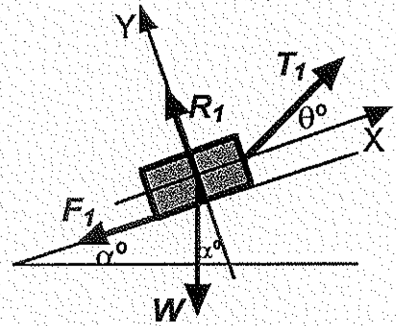
1.a) 10 N; rest b) 14 N; rest c) 14 N; accelerate d) 10 N; rest e) 14 N; rest f) 17·92 N;accelerate g) 10 N; rest h) 10·08 N; accelerate i) 10·08 N; accelerate j) 12·04 N; accelerate k) 12·12 N; rest l) 21·84 N; accelerate 2. $\mu = \frac{4}{7}$ 3. 49 N; moves 4. 3·92 N; 4·62 N; 6·93 N
5. 9·8 N; No motion

Bodies on an inclined plane

The diagrams below show a block on the point of **slipping up** a plane inclined at an angle $\alpha°$ to the horizontal being 'pulled' by a string which, on the left is parallel to the plane and on the right is inclined at an angle $\theta°$ to the plane. The forces acting on the blocks are shown and the required axes have also been indicated.



String parallel to plane



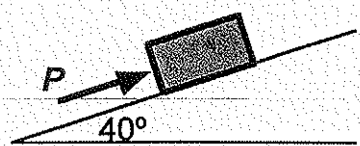
String inclined to the plane

	String parallel to plane	String inclined to the plane
Limiting	so $F = \mu R$ and $\sum \underline{F} = \underline{0}$	$F_1 = \mu R_1$ and $\sum \underline{F} = \underline{0}$
Equilibrium	so $\sum F_X = 0$ and $\sum F_Y = 0$	so $\sum F_X = 0$ and $\sum F_Y = 0$
	Giving $T - F - W\, sin\,\alpha^0 = 0$	Giving $T_1 cos\,\theta^0 - F_1 - W\, sin\,\alpha^0 = 0$
	and $R - W\, cos\,\alpha^0 = 0$	and $R_1 + T_1 sin\,\theta^0 - W\, cos\,\alpha^0 = 0$
	So $T = F + W\, sin\,\alpha^0$	So $T_1 = \dfrac{F_1 + W\, sin\,\alpha^0}{cos\,\theta^0}$
	and $R = W\, cos\,\alpha^0$	and $R_1 = W\, cos\,\alpha^0 - T_1\, sin\,\theta^0$
	Giving $T = \mu R + W\, sin\,\alpha^0$	Giving $T_1 = \dfrac{\mu R_1 + W\, sin\,\alpha^0}{cos\,\theta^0}$
	and $R = mg\, cos\,\alpha^0$	and $R_1 = mg\, cos\,\alpha^0 - T_1\, sin\,\theta^0$

WORKED EXAMPLES

Example 1

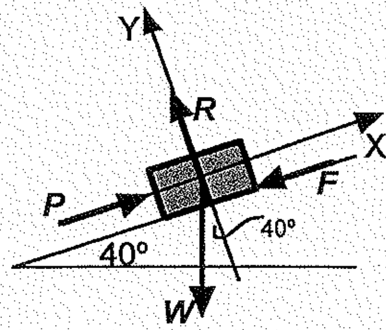
A block of wood of mass 6 kg is placed on a rough slope, inclined at an angle of 40° to the horizontal. It is held in position by the action of a force of P newtons acting parallel to the slope. The coefficient of friction between the block of wood and the surface of the slope is 0·4.



Find the greatest and least values of P for the block to remain stationary on the slope.

Solution

On the point of moving up the slope



On the point of slipping up the slope, so Friction force F acts down the slope.

For limiting equilibrium $\sum F_X = 0$ and $\sum F_Y = 0$ and $F = \mu R$

So $P - F - W\, sin\,40^0 = 0$ and $R - W\, cos\,40^0 = 0$

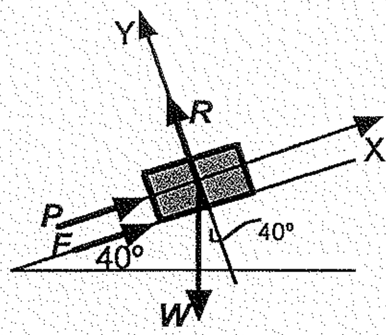
Giving $P = \mu R + mg\, sin\,40^0$ and $R = mg\, cos\,40^0$

So $P = mg(\mu\, cos\,40^0 + sin\,40^0)$

$P = 6 \times 9{\cdot}8 \times (0{\cdot}4 \times cos\,40^0 + sin\,40^0)$

$P \approx 55{\cdot}8$ N

On the point of moving down the slope



On the point of moving down the slope, so Friction force F acts up the slope.

For limiting equilibrium $\sum F_X = 0$ and $\sum F_Y = 0$ and $F = \mu R$

So $P + F - W\sin 40^0 = 0$ and $R - W\cos 40^0 = 0$

Giving $P = mg\sin 40^0 - \mu R$ and $R = mg\cos 40^0$

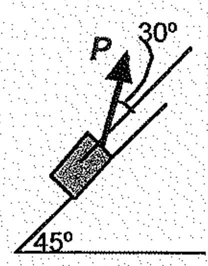
So $P = mg(\sin 40^0 - \mu\cos 40^0)$

$P = 6 \times 9{\cdot}8 \times (\sin 40^0 - 0{\cdot}4 \times \cos 40^0)$

$P \approx 19{\cdot}8$ N

For the block to remain stationary on the plane $19{\cdot}8 \le P \le 55{\cdot}8$.

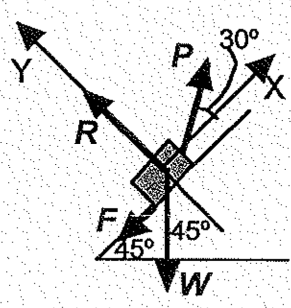
Example 2



A block of mass 4 kg rests on a plane inclined at an angle of 45° to the horizontal, under the action of a force of magnitude P acting upwards at an angle of 30° to the line of greatest slope of the plane.

If the coefficient of friction is $\frac{1}{\sqrt{2}}$, calculate the magnitude of P if the block is on the point of moving up the plane.

Solution



Block is on the point of moving up the slope, so Friction force F acts down the slope.

For limiting equilibrium $\sum F_X = 0$ and $\sum F_Y = 0$ and $F = \mu R$

So $P\cos 30^0 - F - W\sin 45^0 = 0$ and $R + P\sin 30^0 - W\cos 45^0 = 0$

Giving $\dfrac{\sqrt{3}P}{2} = \mu R + \dfrac{mg}{\sqrt{2}}$ and $R = \dfrac{mg}{\sqrt{2}} - \dfrac{P}{2}$

So $\dfrac{\sqrt{3}P}{2} = \dfrac{1}{\sqrt{2}}\left(\dfrac{mg}{\sqrt{2}} - \dfrac{P}{2}\right) + \dfrac{mg}{\sqrt{2}}$

$$\frac{\sqrt{3}P}{2} = \frac{mg}{2} - \frac{P}{2\sqrt{2}} + \frac{mg}{\sqrt{2}}$$

$$\frac{\sqrt{3}P}{2} + \frac{P}{2\sqrt{2}} = mg\left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)$$

$$\left(\frac{\sqrt{6}+1}{2\sqrt{2}}\right)P = 4 \times 9{\cdot}8 \times \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right)$$

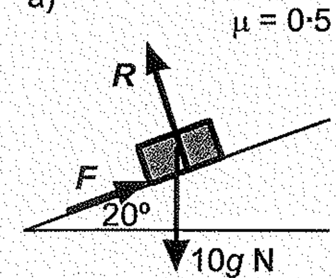
$$P = 4 \times 9{\cdot}8 \times \left(\frac{1}{2} + \frac{1}{\sqrt{2}}\right) \times \frac{2\sqrt{2}}{\sqrt{6}+1}$$

$P \approx 38{\cdot}8$ N

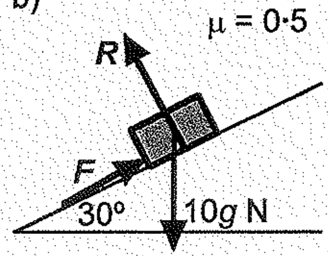
Exercise 1.4-5

1. Each of the following diagrams shows a body of mass 10 kg released from rest on a rough inclined plane. All letters have the same meaning as used in the notes. In each case find the magnitude of the friction force and state whether the body will remain at rest or will begin to slip down the plane.

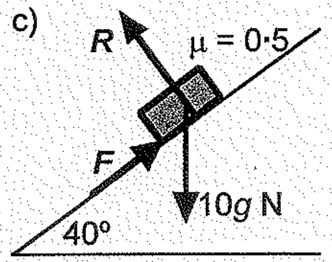
a)



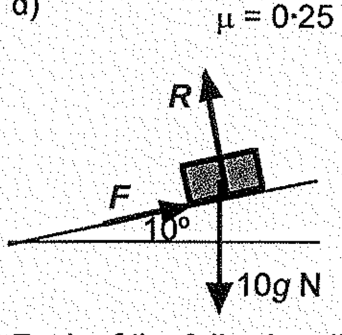
b)



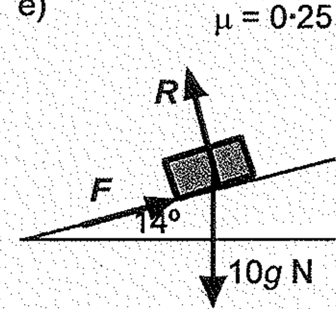
c)



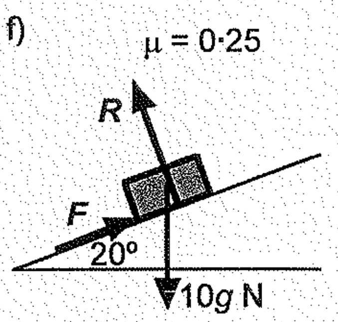
d)



e)

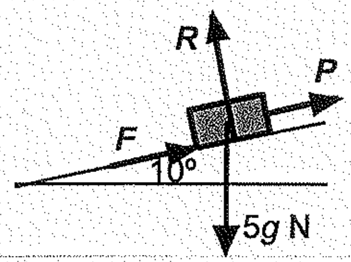


f)

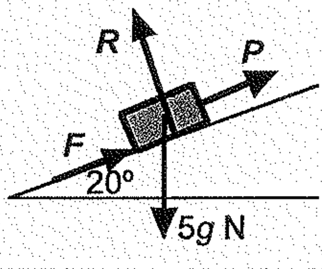


2. Each of the following diagrams shows a body of mass 5 kg on a rough inclined plane with coefficient of friction $\frac{1}{7}$. In each case, find the magnitude of the force P if it just prevents the body from slipping down the plane.

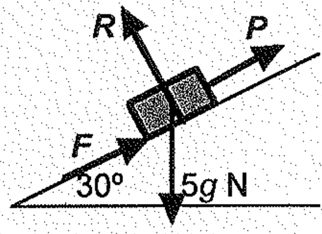
a)



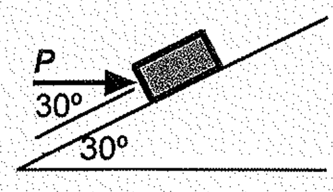
b)



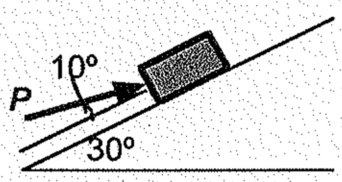
c)



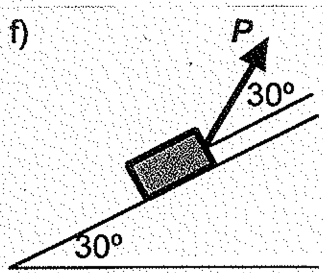
d)



e)



f)



Answers

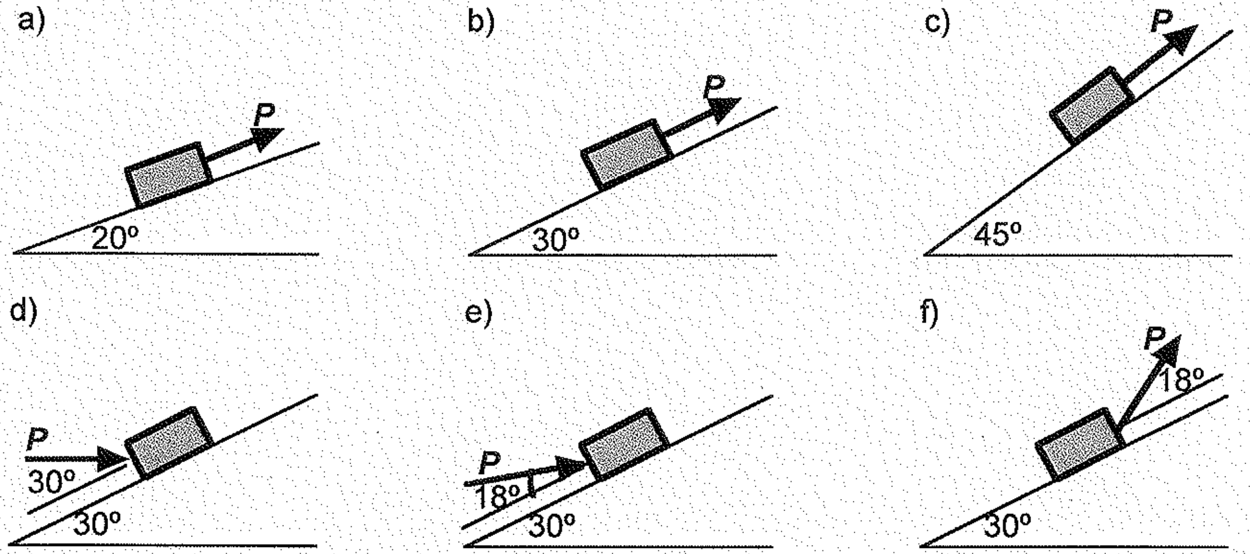
1. a) 33·5 N; rest b) 42·4 N; accelerate c) 37·5 N; accelerate d) 17·0 N; rest e) 23·7 N; rest f) 23·0 N; accelerate
2. a) 1·61 N b) 10·2 N c) 18·4 N d) 19·7 N e) 18·3 N f) 23·2 N

1·62N

Exercise continues on the next page

Exercise 1.4-5 (continued)

3. Each of the following diagrams shows a body of mass 3 kg on a rough inclined plane with coefficient of friction $\frac{1}{3}$. In each case find the magnitude of the force P if the body is just on the point of moving up the plane.



4. A parcel of mass 1 kg is placed on a rough plane which is inclined at 30° to the horizontal. The coefficient of friction between the parcel and the plane is 0·25. Find the force that must be applied to the parcel in a direction parallel to the plane so that
   a) the parcel is prevented from sliding down the plane,
   b) the parcel is on the point of moving up the plane.

5. A horizontal force of 1 N is just sufficient to prevent a brick of mass 600 g sliding down a rough plane which is inclined at $\sin^{-1}\frac{5}{13}$ to the horizontal. Find the coefficient of friction between the brick and the plane.

6. a) A mass m lies on a rough plane which is inclined at an angle θ to the horizontal. The coefficient of friction between the mass and the plane is μ. Show that slipping will occur if $\tan\theta > \mu$.
   b) Will slipping occur when a body is placed on a rough plane with coefficient of friction 0·5 and angle 40° to the horizontal?
   c) Will slipping occur when a body is placed on a rough plane with coefficient of friction 0·25 and angle 10° to the horizontal?
   d) When a body is placed on a rough plane which is inclined at 30° to the horizontal, the body is found to be in limiting equilibrium. Find the coefficient of friction between the body and the plane.

7. A force F acting parallel to and up a rough plane of inclination θ, is just sufficient to prevent a body of mass m from sliding down the plane. A force $4F$ acting parallel to and up the same rough plane causes the mass m to be on the point of moving up the plane. If μ is the coefficient of friction between the mass and the plane, show that $5\mu = 3\tan\theta$.

Answers

3. a) 19·3 N b) 23·2 N c) 27·7 N d) 33·2 N e) 27·3 N f) 22·0 N 4. a) 2·78 N b) 7·02 N
5. 0·23 6. b) Yes c) No d) 0·577

ANGLE OF FRICTION

A special name is given to the situation where an object on an inclined plane is just on the point of slipping. The angle at which this begins to happen is called the **angle of friction**. Consider a block of mass m kilograms at rest on a plane, which is gradually tilted until the block is on the point of moving down the plane. Suppose the angle of the plane to the horizontal is λ when the block is on the point of slipping and the coefficient of friction between the block and the plane is μ.

Since the block is on the point of moving down the plane, friction is acting up the plane.

For limiting equilibrium $\sum F_X = 0$ and $\sum F_Y = 0$ and $F = \mu R$

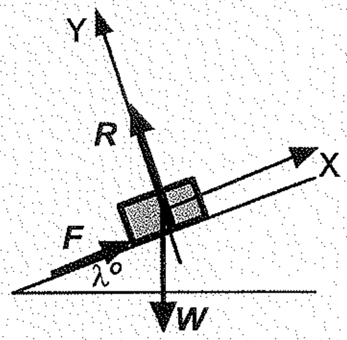
So $F - W\sin\lambda^0 = 0$ and $R - W\cos\lambda^0 = 0$

Giving $\mu R = mg\sin\lambda^0$ and $R = mg\cos\lambda^0$

So $\mu mg\cos\lambda^0 = mg\sin\lambda^0$

$$\mu = \frac{mg\sin\lambda^0}{mg\cos\lambda^0} = \tan\lambda^0$$

λ^0 is known as the angle of friction.



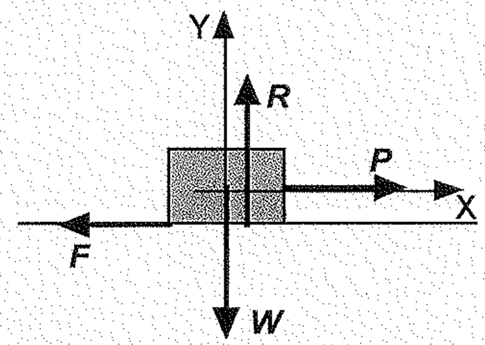
This result offers a simple way to experimentally determine the coefficient of friction between two surfaces.

WORKED EXAMPLES

Example 1

When a horizontal force P of magnitude 20 N is applied to a body of mass 6 kg resting on a rough horizontal surface, the body is in limiting equilibrium. Calculate the angle of friction.

Solution



On point of moving so in limiting equilibrium

$\sum F_X = 0$ and $\sum F_Y = 0$ and $F = \mu R$

So $P - F = 0$ and $R - W = 0$

Giving $P = \mu R$ and $R = mg$

$$\text{So } \mu = \frac{P}{R} = \frac{20}{6 \times 9{\cdot}8} = 0{\cdot}340\ldots$$

Using $\tan\lambda^0 = \mu$ then $\lambda^0 = \tan^{-1}(0{\cdot}340\ldots) \approx 18{\cdot}8^0$

So angle of friction $= 18{\cdot}8^0$.

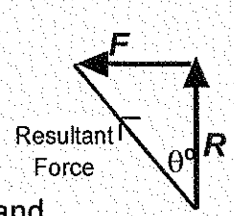
Example 2

A block of mass m kilograms, resting on a horizontal plane, is acted upon by a horizontal force P of magnitude P newtons. Show that if the resultant of the frictional force and the normal reaction make an angle of θ with the vertical then $\tan\theta^0 \le \mu$.

Solution

Using the Free Body diagram above the resultant of R and F is shown below.

Since the block is at rest, $F \le \mu R$.

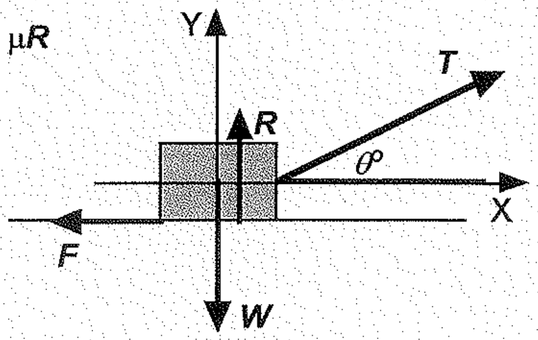


From the vector diagram $\tan\theta^0 = \dfrac{F}{R} \le \dfrac{\mu R}{R}$ So $\tan\theta^0 \le \mu$ and using $\mu = \tan\lambda^0$ then $\tan\theta^0 \le \tan\lambda^0$. It follows that $\theta \le \lambda$. If $\theta < \lambda$, the block remains stationary and when $\theta = \lambda$ it begins to move. This is the basis of an alternative definition of the angle of friction.

Example 3

A block of mass *m* kilograms is lying on a horizontal plane and is acted upon by a force ***T*** of magnitude *T* newtons which makes an angle of θ^o above the horizontal. If the block is on the point of moving, find the value of θ^o such that *T* is a minimum.

Solution



For limiting equilibrium $\sum F_X = 0$ and $\sum F_Y = 0$ and $F = \mu R$

So $T\cos\theta^0 - F = 0$ and $R + T\sin\theta^0 - W = 0$

Giving $T\cos\theta^0 = \mu R$ and $R = mg - T\sin\theta^0$

So $T\cos\theta^0 = \mu mg - \mu T\sin\theta^0$

$T\cos\theta^0 + \mu T\sin\theta^0 = \mu mg$

$T(\cos\theta^0 + \mu\sin\theta^0) = \mu mg$

So $T = \dfrac{\mu mg}{(\cos\theta^0 + \mu\sin\theta^0)}$

Using $\mu = \tan\lambda^0 = \dfrac{\sin\lambda^0}{\cos\lambda^0}$ then $T = \dfrac{\dfrac{\sin\lambda^0}{\cos\lambda^0}mg}{(\cos\theta^0 + \dfrac{\sin\lambda^0}{\cos\lambda^0}\sin\theta^0)}$

Multiplying top and bottom of fraction by $\cos\lambda^0$

Gives $T = \dfrac{\sin\lambda^0 mg}{(\cos\theta^0\cos\lambda^0 + \sin\theta^0\sin\lambda^0)}$ but $\cos\theta^0\cos\lambda^0 + \sin\theta^0\sin\lambda^0 = \cos(\theta - \lambda)^0$

So $T = \dfrac{\sin\lambda^0 mg}{\cos(\theta - \lambda)^0}$. *T* has a minimum value of $\sin\lambda^0 mg$ when $\theta = \lambda$.

This result complements the solution for Example 2(d) on pages 53 to 54.

Exercise 1.4-6

1. The coefficient of friction for two surfaces in contact is 0·2. Find the angle of friction for the two surfaces.
2. The angle of friction for two surfaces in contact is 30°. Find the coefficient of friction for the two surfaces.
3. A body of mass 4 kg lies on a rough plane inclined at 30° to the horizontal. The angle of friction between the plane and the body is 15°. Find the magnitude of the least horizontal force that must be applied to the body to prevent motion down the plane.
4. A body of mass 2 kg lies on a rough plane which is inclined at 40° to the horizontal. The angle of friction between the plane and the body is 15°. Find the magnitude of the greatest force which can be applied to the body, parallel to and up the plane, without motion occurring.
5. Show that the least **pulling** force ***P*** sufficient to ensure that a body is on the point of moving up a plane inclined at θ° to the horizontal is *mg* $\sin(\theta + \lambda)^o$ and that it occurs when ***P*** is inclined at λ^o to the plane.

Answers

1. 11·3° 2. 0·577 3. 10·5 N 4. 16·6 N

Outcome Content
Know Newton's second law of motion; that force is the rate of change of momentum, and derive the equation $\underline{F} = m\underline{a}$.
Use the above equation to form equations of motion to model practical problems on motion in a straight line.
Solve such equations modelling motion in one dimension, including cases where acceleration is dependent on time.

Deriving the equation $\underline{F} = m\underline{a}$

Newton's Second Law of Motion states that the rate of change of momentum of a body is directly proportional to the applied force and is in the direction of that force. The **momentum** of a body of mass m kilograms and velocity $\underline{v}$ is the vector quantity $m\underline{v}$. If it is acted upon by a force $\underline{F}$ then:

From Newton's Second Law $\underline{F} \propto \dfrac{d(m\underline{v})}{dt}$

So $\underline{F} = k\dfrac{d(m\underline{v})}{dt}$, where k is the constant of variation.

If the mass remains constant then $\dfrac{d(m\underline{v})}{dt} = m\dfrac{d\underline{v}}{dt} = m\underline{a}$.

So $\underline{F} = km\underline{a}$

The unit of force, the newton, is chosen such that a force of 1 newton acting on a mass of 1 kilogram will cause an acceleration of 1 ms^{-2}, so $k = 1$ and hence $\underline{F} = m\underline{a}$.

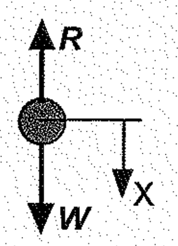
WORKED EXAMPLES

Example 1

A pile of mass 6000 kg starts to penetrate the ground with a speed of 7 ms^{-1} and penetrates the ground to a depth of 25 centimetres before it is brought to rest. Calculate the resistance of the ground to the penetration, assuming this resistance is constant.

Solution

Free Body Diagram



The force applied to the pile is $W - R$ and since this is constant the acceleration / deceleration is also constant. So the equation of motion $v^2 = u^2 + 2as$ can be used.

Here $v = 0$, $u = 7$ ms^{-1}, and $s = 0{\cdot}25$ m

So $0 = 7^2 + 2a \times 0{\cdot}25$

Giving $a = -\dfrac{49}{0{\cdot}5} = -98$ ms^{-2}

Using Newton's Second Law $F = ma$

$$W - R = ma$$

$$6000g - R = 6000 \times (-98)$$

$$R = 6000 \times 9{\cdot}8 + 6000 \times 98$$

$$R = 646800 \text{ N}$$

Example 2

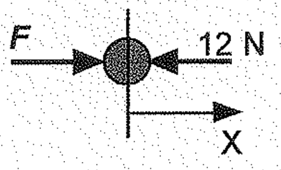
A body of mass 4 kilograms moves along a horizontal straight line under the action of two forces in the direction of the line. One force is a variable propulsive force of magnitude *F* newtons and the other is a constant resistive force of magnitude 12 newtons. The body starts from rest and acquires a speed of *v* ms^{-1}, in time *t* seconds, given by $v = 10t - t^2$ for $0 \le t \le 6$.

a) Find an expression for *F* in terms of *t*.

b) Calculate the value of *F* when the body has reached maximum speed.

Solution

Free Body Diagram



a) Using Newton's Second Law $\sum F_x = ma$

So $F - 12 = 4a$

Now $v = 10t - t^2$ and $a = \dfrac{dv}{dt} = 10 - 2t$

So $F - 12 = 4(10 - 2t)$

Giving $F = 52 - 8t$

b) At maximum speed when $a = 0$ or $\sum F_x = 0$ so $F = 12$ N.

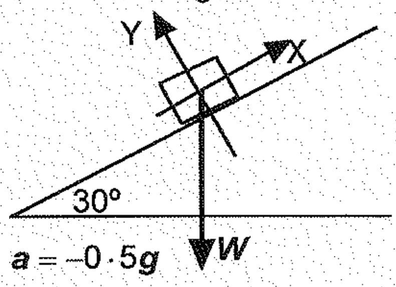
Example 3

A block is projected with speed 7 ms^{-1} up a **smooth** plane which is inclined at 30° to the horizontal. Find the distance it travels before coming to rest momentarily.

Solution

The block is decelerated by its weight component which acts down the slope. As this is constant the equations of motion can be used.

Free Body Diagram



Applying Newton's Second Law in the direction of the motion.

Gives $\sum F_x = ma$

So $-W \sin 30^0 = ma$

and $-mg \sin 30^0 = ma$

Giving $a = -0{\cdot}5g$

Using the equation of motion $v^2 = u^2 + 2as$ where $u = 7$, $v = 0$ and $a = -0{\cdot}5g$

Gives $0 = 7^2 + 2 \times (-0{\cdot}5 \times 9{\cdot}8) \times s$

$$s = \frac{49}{9{\cdot}8} = 5 \text{ metres}$$

Example 4

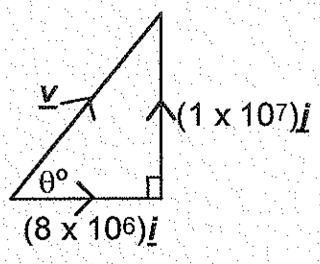
An electron of mass 9 x 10^{-31} kilograms is moving at 8 x 10^6 ms^{-1} when it enters an electric field which produces a force of 1·8 x 10^{-15} N at right angles to its initial velocity for a period of 5 x 10^{-9} seconds. Find its final velocity.

Solution

Let the unit vectors $\underline{i}$ and $\underline{j}$ be in the directions of the initial velocity and the force respectively, then $\underline{u} = (8 \times 10^6)\underline{i}$ and, from Newton's Second Law, $\underline{F} = m\underline{a}$ gives:

$$(1{\cdot}8 \times 10^{-15})\underline{j} = (9 \times 10^{-31})\underline{a}$$

So $$\underline{a} = \frac{(1{\cdot}8 \times 10^{-15})}{(9 \times 10^{-31})}\underline{j} = (2 \times 10^{15})\underline{j}$$



Using $\underline{v} = \underline{u} + t\underline{a}$ gives $\underline{v} = (8\times10^{6})\underline{i} + (2\times10^{15})t\underline{j}$

When $t = 5\times10^{-9}$ $\underline{v} = (8\times10^{6})\underline{i} + (2\times10^{15})\times(5\times10^{-9})\underline{j}$

$\underline{v} = (8\times10^{6})\underline{i} + (1\times10^{7})\underline{j}$

So $|\underline{v}|^2 = (8\times10^{6})^2 + (1\times10^{7})^2 = 164\times10^{12}$ and $\tan\theta^0 = \dfrac{1\times10^{7}}{8\times10^{6}} = 1{\cdot}25$

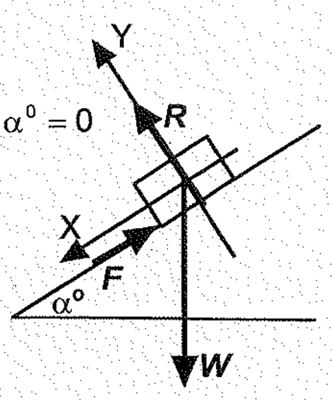
Giving $|\underline{v}| = \sqrt{164\times10^{12}} \approx 1{\cdot}28\times10^{7}$ ms^{-1} at an angle of $\tan^{-1}1{\cdot}25 \approx 51{\cdot}3^0$ to the original direction as shown.

Example 5

A block of mass *m* kg is sliding down a plane inclined at an angle of α^o to the horizontal. If the coefficient of **dynamic** friction between the block and the plane is μ find the acceleration of the block.

Solution

The forces acting on the block are its weight, the normal reaction to the plane and friction. Friction opposes the blocks motion and acts up the slope.



The block is in motion so friction has its limiting value, $F = \mu R$.

There is no motion in the Y direction so $\sum F_Y = 0$ which gives $R - W\cos\alpha^0 = 0$

So $R = mg\cos\alpha^0$ which gives $F = \mu mg\cos\alpha^0$

Using Newton's Second Law in the X direction gives, $\sum F_X = ma$

So $W\sin\alpha^0 - F = ma$ which gives $mg\sin\alpha^0 - \mu mg\cos\alpha^0 = ma$

So $a = g(\sin\alpha^0 - \mu\cos\alpha^0)$ ms^{-2}

Exercise 1.4-7

1. Find, in vector form, the resultant force required to make a body of mass 2 kilograms accelerate at $(5\underline{i} + 2\underline{j})$ ms^{-2}.
2. Find, in vector form, the acceleration produced in a body of mass 500 grams subject to the forces $(4\underline{i} + 2\underline{j})$ N and $(-\underline{i} + \underline{j})$ N.
3. A car travels a distance of 24 metres whilst uniformly accelerating from rest to 12 ms^{-1}. Find the acceleration of the car.
   If the car has a mass of 600 kilograms find the magnitude of the accelerating force.
4. A car moves along a level road at a constant velocity of 22 ms^{-1}. If its engine is exerting a forward force of 500 N, what resistance is the car experiencing?
5. A car of mass 500 kilograms moves along a level road with an acceleration of 2 ms^{-2}. If its engine is exerting a forward force of 1100 N, what resistance is the car experiencing?
6. Find, in vector form, the acceleration produced in a body of mass 500 grams when forces of $(5\underline{i} + 3\underline{j})$ N, $(6\underline{i} + 4\underline{j})$ N and $(-7\underline{i} - 7\underline{j})$ N act on the body.
7. A train of mass 60 tonnes is travelling at 40 ms^{-1} when the brakes are applied. If the resultant braking force is 40 kN, find the distance the train travels before coming to rest.

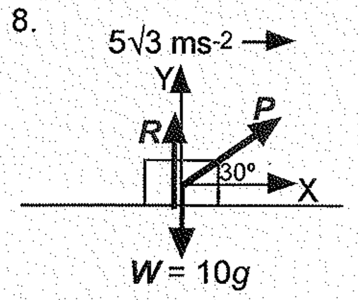
Answers

1. $(10\underline{i} + 4\underline{j})$ N	2. $(6\underline{i} + 6\underline{j})$ ms^{-2}	3. 3 ms^{-2}; 1800 N	4. 500 N
5. 100 N	6. $8\underline{i}$ ms^{-2}	7. 1200 m	

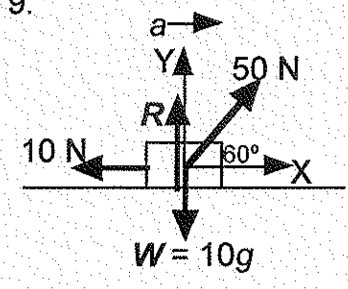
Exercise 1.4-7 continued

Each of the diagrams in question 8 to 12 shows a body of mass 10 kilograms accelerating along a surface in the direction indicated. All of the forces acting are shown and the surfaces are **smooth**. In each case
a) obtain an equation connecting forces acting perpendicular to the direction of motion,
b) obtain an equation by applying Newton's second law to the direction of motion,
c) use your results from a) and b) to establish any unknown forces, accelerations or angles.

8.



9.



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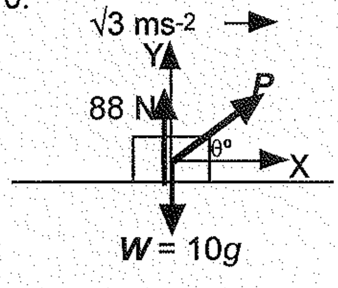

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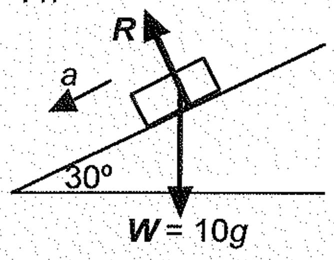
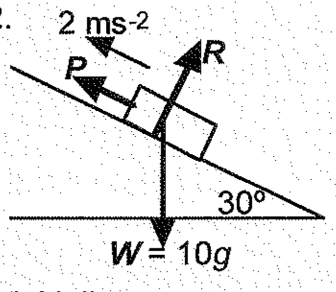


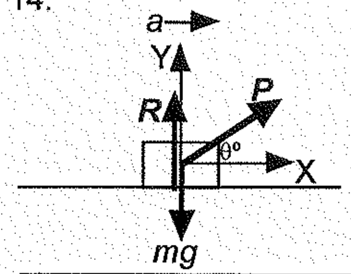
12.



13. A body of mass 10 kilograms is initially at rest on a rough horizontal surface. It is pulled along the surface with constant force 60 N inclined at 60° above the horizontal. If the resistance to motion totals 10 N, find the acceleration of the body and the distance travelled in the first 3 seconds.

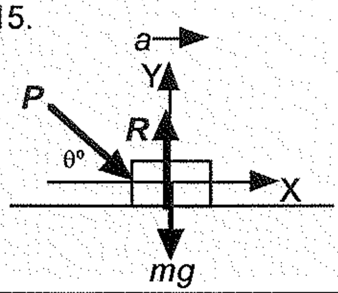
Each of the diagrams in questions 14 to 16 shows a mass accelerating in the direction indicated. The surfaces in contact are **smooth**.

14.



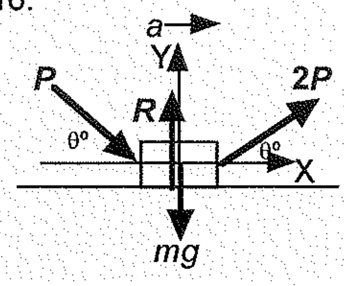
Prove that $\tan\theta^0 = \dfrac{mg - R}{ma}$

15.



Prove that $\tan\theta^0 = \dfrac{R - mg}{ma}$

16.



Prove that $3\boldsymbol{R} = \boldsymbol{m}(3\boldsymbol{g} - \boldsymbol{a}\tan\theta^0)$

17. A body of mass m is pulled along a smooth horizontal surface by a force P inclined at $\theta°$ above the horizontal. If the mass starts from rest, show that the distance moved in time t is given by $\dfrac{Pt^2\cos\theta^0}{2m}$.

Answers

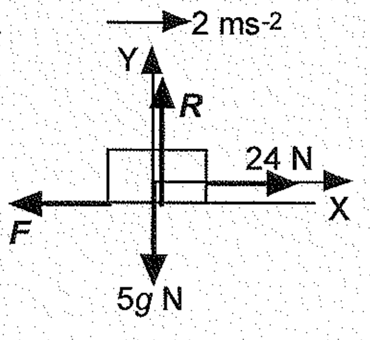
8. $P = 100$ N; $R = 48$ N
9. $R = 54{\cdot}7$ N; $a = 1{\cdot}5$ ms^{-2}
10. $P = 20$ N; $\theta = 30°$
11. $R = 84{\cdot}9$ N; $a = 4{\cdot}9$ ms^{-2}
12. $R = 84{\cdot}9$ N; $P = 69$ N
13. 2 ms^{-2}; 9 m

Exercise 1.4-7 continued

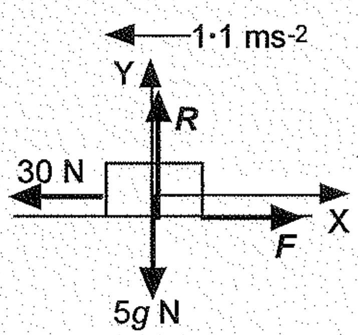
18. A body of mass m is pulled along a rough horizontal surface by a force P inclined at θ^o above the horizontal. If the mass accelerates from rest to velocity v in a distance d, show that the resistance to motion, assumed constant, is $P\cos\theta^0 - \dfrac{mv^2}{2d}$.
19. A mass of 5 kilograms is pulled along a rough horizontal surface by a force of 50 N inclined at 60° above the horizontal. The mass starts from rest and after 4 seconds the pulling force ceases. If the resistance to motion is 20 N throughout, find the total distance travelled before the mass comes to rest again.

In questions 20 to 25 the forces shown cause the body of mass 5 kilograms to accelerate along the rough horizontal plane. In each question find the coefficient of friction.

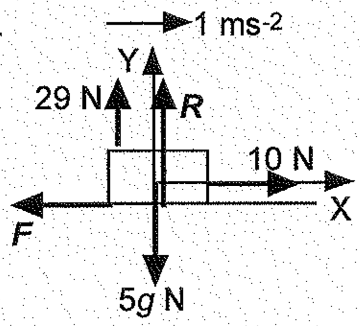
20.



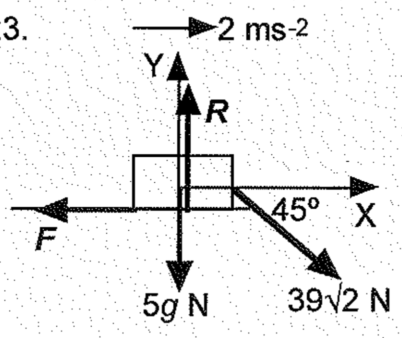
21.



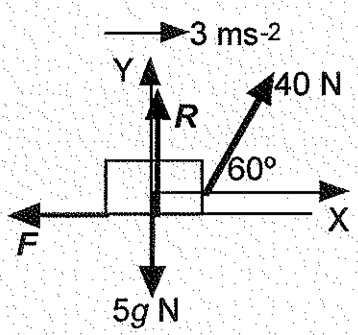
22.



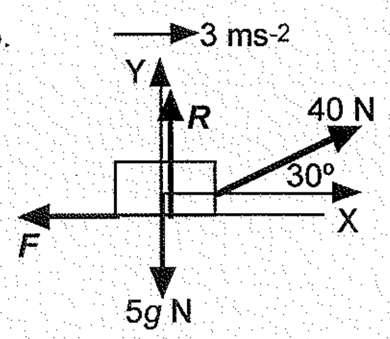
23.



24.



25.



26. A block of mass 20 kilograms rests on a rough horizontal plane. The coefficient of friction between the block and the plane is 0·25. Calculate the frictional force experienced by the block when a horizontal force of 50 N acts on the block. State whether the block will move and if so, find its acceleration.
27. A block of mass 2 kilograms is initially at rest on a rough horizontal surface. The coefficient of friction between the block and the surface is 0·5. Find the horizontal force that must be applied to the block to cause it to accelerate along the surface at 5 ms^{-2}.
28. When a horizontal force of 37 N is applied to a body of mass 10 kilograms which is resting on a rough horizontal surface, the body moves along the surface with an acceleration of 1·25 ms^{-2}. Find μ, the coefficient of friction between the body and the surface.
29. A body of mass 2 kilograms is sliding along a **smooth** horizontal surface at a constant speed of 2 ms^{-1} when it encounters a **rough** horizontal surface, coefficient of friction 0·2. Find the distance that the body will move across the rough surface before coming to rest.

Answers

19. 10 m 20. $\frac{2}{7}$ 21. 0·5 22. 0·25 23. 0·33 24. 0·348 25. 0·677
26. 49 N; Yes; 0·05 ms^{-2} 27. 19·8 N 28. 0·25 29. 1·02 m

Exercise 1.4-7 continued

30. A horizontal force F of constant direction and variable magnitude given by $F = 2t$ acts on a body of mass 5 kilograms. The body is initially at rest at an origin O on a **smooth** horizontal surface. Find the velocity of the body when $t = 4$ seconds.

31. A horizontal force F of constant direction and variable magnitude given by $F = 6t - 4$ acts on a body of mass 2 kilograms. The body is initially at rest at an origin O on a **smooth** horizontal surface. Find the displacement of the body when $t = 4$ seconds.

32. A horizontal force F of constant direction and variable magnitude given by $F = 3t + 1$ acts on a body of mass 4 kilograms. The body is initially at rest at an origin O on a **smooth** horizontal surface. Find
a) the velocity of the body when $t = 2$ seconds
b) the displacement of the body when $t = 2$ seconds.

33. A body of mass 750 grams is initially at rest at a point O on a smooth horizontal surface. A horizontal force F acts on the body and causes it to move in a straight line across the surface. The magnitude of the force is given by $F = (3t + 1)$ N where t is the time in seconds from the start of the motion. Find the speed of the body when $t = 3$ and its distance from O at that time.

34. A resultant force $\underline{F}$ acts on a body of mass 500 grams initially at rest at an origin O. $\underline{F} = (3t\underline{i} + \underline{j})$ N where t seconds is the time for which the force has been acting on the body. Find expressions for
a) the velocity vector of the body at time t
b) the position vector of the body at time t.

35. A resultant force $\underline{F}$ acts on a body of mass 250 grams initially at rest at an origin O. $\underline{F} = [(5t - 2)\underline{i} + 4t\underline{j}]$ N where t seconds is the time for which the force has been acting on the body. Find expressions for
a) the velocity vector of the body at time t
b) the position vector of the body at time t.

36. A resultant force $\underline{F}$ acts on a body of mass 500 grams initially at rest at an origin O. $\underline{F} = [(4t - 1)\underline{i} + 4\underline{j}]$ N where t seconds is the time for which the force has been acting on the body. Find the body's speed and distance from O when $t = 2$ seconds.

Answers

30. 3·2 ms^{-1} in the direction of F 31. 16 m in the direction of F
32. a) 2 ms^{-1} in the direction of F b) 1·5 m in the direction of F
33. a) 22 ms^{-1} b) 24 m 34. a) $\underline{v} = (3t^2\underline{i} + 2t\underline{j})$ ms^{-1} b) $\underline{r} = (t^3\underline{i} + t^2\underline{j})$ m
35. a) $\underline{v} = \left[(10t^2 - 8t)\underline{i} + 8t^2\underline{j}\right]$ ms^{-1} b) $\underline{r} = \left[(\frac{10}{3}t^3 - 4t^2)\underline{i} + \frac{8}{3}t^3\underline{j}\right]$ m
36. 20 ms^{-1}; $17\frac{1}{3}$ m